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Lecture 5

Integration of Complex Functions:

As before μ is a positive measure on a measurable space X.

Definition: $L^1(\mu)$ is the set of complex m'ble functions f on X such that

$$\int_X |f|\, d\mu < \infty.$$

Recall, m'bility of $f \Rightarrow$ m'bility of $|f|$, and hence $\int_X |f|\, d\mu$ makes sense as an element of $[0,\infty]$.

Elements of $L^1(\mu)$ are called Lebesgue integrable functions or, more often, simply as <u>integrable functions</u> (w.r.t. μ). They are also called <u>summable</u> functions.

Definition: If $f = u + iv$ where u and v are real m'ble functions on X, and if $f \in L^1(\mu)$, define

$$\int_E f\, d\mu = \int_E u^+\, d\mu - \int_E u^-\, d\mu + i\left\{ \int_E v^+\, d\mu - \int_E v^-\, d\mu \right\}$$

for every m'ble set E.

Since u^+, u^-, v^+, v^- are real, m'ble, non-negative, therefore the four integrals on the right exist. Moreover, all four are less than or equal to $|f|$, and $\int_E |f|\, d\mu < \infty$. Hence all four integrals are finite and no problems occur when subtracting two of them. Thus $\int_E f\, d\mu \in \mathbb{C}$.

Remark: Sometimes it is useful to define $\int_X f\,d\mu$ when f is m'ble and takes values in $[-\infty,\infty]$. To do this we have to assume at least one of $\int_E f^+ d\mu$ or $\int_E f^- d\mu$ is finite. If this is so, one sets

$$\int_E f\,d\mu = \int_E f^+ d\mu - \int_E f^- d\mu.$$

This is a number in $[-\infty,\infty]$.

Theorem: $L^1(\mu)$ is a complex vector space and $\int_X (\text{---})\,d\mu$ is a complex linear functional on $L^1(\mu)$.

Proof: We have already seen that if f, g are complex m'ble then so is $\alpha f + \beta g$ for every $\alpha, \beta \in \mathbb{C}$. Suppose $f, g \in L^1(\mu)$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\int_X |\alpha f + \beta g|\,d\mu \le \int_X \{|\alpha|\,|f| + |\beta|\,|g|\}\,d\mu$$

$$= |\alpha| \int_X |f|\,d\mu + |\beta| \int_X |g|\,d\mu < \infty.$$

Thus $\alpha f + \beta g \in L^1(\mu)$, whence $L^1(\mu)$ is a complex vector space.

In order to prove

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$$

for f, g in $L^1(\mu)$, it is enough to prove this when f, g are real valued by definition of a $\int_X (\ )\,d\mu$ on complex functions in $L^1(\mu)$.

So assume $f, g \in L^1(\mu)$ and are real valued. Set

$$h = f + g.$$

Then

$$h^+ + f^- + g^- = f^+ + g^+ + h^-.$$

We have seen that $\int_X (-)\,d\mu$ respects addition for $[0,\infty)$ valued functions. Since all the terms occurring on both sides of the above relation are $[0,\infty)$ valued we have

$$\int_X h^+\,d\mu + \int_X f^-\,d\mu + \int_X g^-\,d\mu = \int_X f^+ d\mu + \int_X g^+ d\mu + \int_X h^-\,d\mu.$$

From here one sees easily that

$$\int_X h\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$$

as required.

It remains to show that

(*) $$\int_X \alpha f\,d\mu = \alpha \int_X f\,d\mu \quad \text{for } \alpha \in \mathbb{C},\ f \in L^1(\mu).$$

Recall that

$$\int_X cg\,d\mu = c\int_X g\,d\mu$$

if $c \in [0,\infty)$ and $g \geq 0$.

From the above it is easy to see from the definitions that (*) holds for $\alpha \geq 0$. Next, using relations like $(-g)^+ = g^-$ and $(-g)^- = g^+$ and breaking up $f \in L^1(\mu)$ as $f = u + iv$, with u, v real valued, that (*) holds for $\alpha \in \mathbb{R}$. Finally (with f in $L^1(\mu)$ and u, v the real and imaginary parts of f) we have

$$\begin{aligned}
\int_X (if)\,d\mu &= \int_X v^-\,d\mu - \int_X v^+ d\mu + i\left\{\int_X u^+ d\mu - \int_X u^-\,d\mu\right\} \\
&= i\left[\int_X u^+ d\mu - \int_X u^- d\mu + i\left\{\int_X v^+ d\mu - \int_X v^- d\mu\right\}\right] \\
&= i\int_X f\,d\mu.
\end{aligned}$$

Thus (*) holds for $\alpha = i$. It is now clear that (*) holds for all $\alpha \in \mathbb{C}$ and all $f \in L^1(\mu)$. q.e.d.

Theorem: Suppose $f \in L^1(\mu)$, then

$$\left|\int_X f\, d\mu\right| \le \int_X |f|\, d\mu.$$

Proof:

Let $f \in L^1(\mu)$ and $z = \int_X f\, d\mu$. If $z = 0$ there is nothing to prove. Otherwise set $\alpha = \bar{z}/z$. Then $|\alpha| = 1$ and $\alpha z = |z|$. Let $u = \mathrm{Re}(\alpha f)$. Then

$$\left|\int_X f\, d\mu\right| = |z| = \alpha z = \alpha \int_X f\, d\mu = \int_X \alpha f\, d\mu = \int_X u\, d\mu$$
$$\le \int_X |f|\, d\mu.$$

The equality $\int_X \alpha f\, d\mu = \int_X u\, d\mu$ follows from the fact that $\int_X \alpha f\, d\mu$ is real (in fact equal to $|z|$). The last inequality holds because $|f| - u \geq 0$

q.e.d.

Theorem (Lebesgue's Dominated Convergence Theorem or "DCT"): Suppose $\{f_n\}$ is a sequence of complex m'ble functions on X s.t. $\lim_{n\to\infty} f_n(x) = f(x)$ exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ s.t. $|f_n| \le g$ $\forall\, n \in \mathbb{N}$, then $f \in L^1(\mu)$ and the following hold:

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu \qquad \text{———} \quad (*)$$

and

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0 \qquad \text{———} \quad (**)$$

Proof:

Since $|f_n| \le g$ $\forall\, n$, it follows that $f_n \in L^1(\mu)$ $\forall\, n$.

Since $\left|\int_X (f_n - f)\, d\mu\right| \le \int_X |f_n - f|\, d\mu$, clearly $(**)$ implies $(*)$.

To prove $(**)$ apply Fatou's Lemma to $2g - |f_n - f|$ to get

$$\int_X 2g \, d\mu = \int_X \liminf_{n\to\infty} (2g - |f_n - f|) \, d\mu$$

$$\leq \liminf_{n\to\infty} \int_X (2g - |f_n - f|) \, d\mu$$

$$= 2g + \liminf_{n\to\infty} \left(-\int_X |f_n - f| \, d\mu\right)$$

$$= 2g - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu$$

This implies

$$\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0.$$

Thus

$$0 \leq \liminf_{n\to\infty} \int_X |f_n - f| \, d\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0.$$

Thus

$$\liminf_{n\to\infty} \int_X |f_n - f| \, d\mu = \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu = 0,$$

establishing (**).

q.e.d.

A basic custom in measure theory: If f and g are functions such that $\mu\{f \neq g\} = 0$, then $f \in L^1(\mu) \iff g \in L^1(\mu)$ since

$$\int_E \{|f| - |g|\} \, d\mu \leq \int_E |f - g| \, d\mu = \int_{E \cap \{f = g\}} |f - g| \, d\mu + \int_{E \cap \{f \neq g\}} |f - g| \, d\mu$$

$$= 0 + 0 = 0$$

for all $E \cap \mathfrak{M}$. In general we do not distinguish between such f and g and write $f = g$ a.e. $[\mu]$ (a.e. = "almost everywhere"), and f and g are said to be equal almost everywhere w.r.t. μ. If we say $f \sim g$ if $f = g$ a.e.$[\mu]$, then $\sim$ is clearly an equivalence relation. It is custom not to distinguish between $L^1(\mu)$ and $L^1(\mu)/\sim$. Technically, what we defined as $L^1(\mu)$ is NOT $L^1(\mu)$. $L^1(\mu)$, technically, is

$$\left\{ f \;\middle|\; \int_X |f| \, d\mu < \infty \right\} \Big/ \sim .$$

Similarly for $L^p(\mu)$ which we will define later. It is only after quotienting by $\sim$ do $L^p(\mu)$ become Banach spaces. However it is customary to say that a function f lies in $L^1(\mu)$ (or $L^p(\mu)$) rather than its equivalence class. We will follows this convention since it is a harmless imprecision.