Theorem: If $f_n : X \longrightarrow [0,\infty]$ is m'ble for $n \in \mathbb{N}$ and
$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in X),$$
then
$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu.$$

Proof:

Let $\{s_n\}$ and $\{t_n\}$ be simple m'ble functions s.t. $0 \le s_n, t_n$ and $s_n \uparrow f_1$ and $t_n \uparrow f_2$. We know such sequences exist. Further, we have shown that $\int_X (s_n + t_n)\,d\mu = \int_X s_n\,d\mu + \int_X t_n\,d\mu$. Since $0 \le s_1 + t_1 \le s_2 + t_2 \le \dots$ and $s_n + t_n \longrightarrow f_1 + f_2$, from the monotone convergence theorem we see that
$$\int_X (f_1 + f_2)\,d\mu = \int_X f_1\,d\mu + \int_X f_2\,d\mu.$$
By induction if $g_N = f_1 + f_2 + \dots + f_N$, then
$$\int_X g_N\,d\mu = \sum_{n=1}^{N} \int_X f_n\,d\mu$$

Now $g_N \longrightarrow f$ and $0 \le g_1 \le g_2 \le \dots$. Hence by the MCT (the "monotone convergence theorem")
$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu$$
as required.

q.e.d.

Remarks

1. Recall that a sequence $\{a_n\}$ can be regarded as a function on $\mathbb{N}$. If $f = \{a_n\}$ is a sequence with $a_n \in [0,\infty]$, $n \in \mathbb{N}$,

and μ is the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ then

$$\int_X f\, d\mu = \sum_{n=1}^{\infty} a_n .$$

2. The above theorem translates to a well-known theorem about double series of non-negative extended numbers, namely:

Corollary: If $a_{ij} \geq 0$ for $i, j \in \mathbb{N}$, then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} .$$

Theorem (Fatou's lemma): If $f_n : X \longrightarrow [0,\infty]$ is m'ble for each $n \in \mathbb{N}$, then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu .$$

Proof:

For $k \in \mathbb{N}$ define

$$g_k = \inf_{n \geq k} f_n .$$

Let

$$g = \liminf_{n\to\infty} f_n$$

Then $g_k : X \longrightarrow [0,\infty]$ is m'ble. Moreover

(a) $0 \leq g_1 \leq g_2 \leq \dots$ and $g_k \longrightarrow g$ as $k \to \infty$

(b) $g_k \leq f_n \quad \forall \; n \geq k, \; k \geq 1$

From (b) we see that for $k \geq 1$,

$$\int_X g_k \, d\mu \leq \int_X f_n \, d\mu \quad \forall \ n \geq k,$$

whence $$\int_X g_k \, d\mu \leq \inf_{n \geq k} \int_X f_n \, d\mu.$$

The right side is an increasing sequence in k whose limit as $k \to \infty$ is by definition $\liminf_{n\to\infty} \int_X f_n \, d\mu$. Thus letting $k \to \infty$, and using the MCT for $\{g_k\}$ we get

$$\int_X g \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.$$

q.e.d.

Theorem: Suppose $f : X \longrightarrow [0,\infty]$ is m'ble, and

$$\varphi(E) = \int_E f \, d\mu \qquad (E \in \mathcal{M}).$$

Then φ is a measure on $\mathcal{M}$, and

$$\int_X g \, d\varphi = \int_X gf \, d\mu$$

for every m'ble g on X with range $[0,\infty]$.

Proof:

Let $E_1, E_2, E_3, \ldots$ be a sequence of disjoint members of $\mathcal{M}$, and let $E = \bigcup_n E_n$.

Then

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f.$$

From an earlier theorem, this means

$$\int_X \chi_E f \, d\mu = \sum_{j=1}^{\infty} \int_X \chi_{E_j} f \, d\mu$$

whence

$$\int_E f\,d\mu = \sum_{j=1}^{\infty} \int_{E_j} f\,d\mu.$$

This means $\quad \phi(E) = \sum_{j=1}^{\infty} \phi(E_j).$

Moreover, clearly $\phi(\emptyset) = 0$. Hence ϕ is a positive measure on $\mathcal{M}$.

Suppose s is a simple m'ble function on X, taking values in $[0,\infty)$. Writing

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

with $\alpha_1, \ldots, \alpha_n$ distinct non-negative real numbers, and A_i m'ble, we see from earlier results that

$$\int_X s\,d\phi = \sum_{i=1}^{n} \alpha_i \,\phi(A_i)$$

$$= \sum_{i=1}^{n} \alpha_i \int_{A_i} f\,d\mu$$

$$= \sum_{i=1}^{n} \alpha_i \int_X \chi_{A_i} f\,d\mu$$

$$= \int_X \Big(\sum_{i=1}^{n} \alpha_i \chi_{A_i}\Big) f\,d\mu$$

$$= \int_X s f\,d\mu.$$

Now suppose g is as in the statement of the theorem. Pick a sequence of m'ble simple functions $\{s_n\}$ with $0 \le s_1 \le s_2 \le \ldots$, $s_n \to g$ as $n \to \infty$. Then we have by MCT (applied more than once)

$$\int_X g \, d\varphi = \lim_{n\to\infty} \int_X s_n \, d\varphi$$

$$= \lim_{n\to\infty} \int_X s_n f \, d\mu$$

$$= \int_X g f \, d\mu .$$

The last equality is by the MCT applied to the sequence $\{s_n f\}$. Note that $0 \le s_1 f \le s_2 f \le \ldots$ and $\lim_{n\to\infty} s_n f = gf$.

q.e.d.