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Lecture 4

Conventions

$$a+\infty = \infty + a = \infty \quad \text{if} \quad 0 \le a \le \infty$$

$$a\cdot\infty = \infty\cdot a = \begin{cases} \infty & \text{if } 0 < a \le \infty \\ 0 & \text{if } a = 0 \end{cases}$$

Check that if $a_n \uparrow a$, $b_n \uparrow b$, $a_n, b_n \ge 0 \ \forall n$, then $a_n b_n \longrightarrow ab$.

This means that sums and products of measurable functions into $[0,\infty]$ are measurable.

Integration of positive functions

Throughout this section μ is a positive measure on a m'ble space $(X, \mathcal{M})$.

Definition: Let $s : X \longrightarrow [0,\infty)$ be a m'ble simple function.
Write (note ∞ is excluded)

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $\alpha_1, \dots, \alpha_n$ are the distinct values of s.

Let $E \in \mathcal{M}$. Define

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap E).$$

the convention $0\cdot\infty = 0$ used for it may happen that $\alpha_i = 0$ and $\mu(A_i \cap E) = \infty$

If

$$f : X \longrightarrow [0, \infty] \leftarrow \infty \text{ is included.}$$

is m'ble and $E \in \mathcal{M}$, define

$$\int_E f \, d\mu = \sup \int_E s \, d\mu,$$

the supremum being taken over simple m'ble functions s such that $0 \le s \le f$.

$\int_E f \, d\mu$ is the integral of f over E w.r.t. μ.

Remark: If $f : X \longrightarrow [0, \infty)$ is m'ble and simple, we have two definitions of $\int_E f \, d\mu$ — the original one with $\int_E f d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap E)$ where $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, $\alpha_1, \dots, \alpha_n$ the distinct values of f, and the defn $\int_E f \, d\mu = \sup_{0 \le s \le f} \int_E s \, d\mu$ with s simple. The two definitions coincide because $f \in \{ s \mid s \text{ simple } 0 \le s \le f \}$

Immediate consequences

Suppose $E, A, B \in \mathcal{M}$, and f, g are m'ble functions on X taking values in $[0, \infty]$. Then the following are immediate consequences of the definition of $\int_E (\ ) \, d\mu$ and the properties of μ and $\mathcal{M}$ established earlier:

(a) $f \le g \Rightarrow \int_E f \, d\mu \le \int_E g \, d\mu$

(b) $A \subseteq B \Rightarrow \int_A f \, d\mu \le \int_B f \, d\mu$

(c) c constant $0 \le c < \infty \Rightarrow \int_E c f \, d\mu = c \int_E f \, d\mu$.

(d) $f|_E = 0 \Rightarrow \int_E f\,d\mu = 0$, even if $\mu(E) = \infty$

(e) $\mu(E) = 0 \Rightarrow \int_E f\,d\mu = 0$ even if $f|_E \equiv \infty$.

(f) $\int_E f\,d\mu = \int_X \chi_E f\,d\mu$.

(We emphasize once again that in (a) — (e) above, $f, g \geq 0$.)

The last result shows that we could have defined $\int_X f\,d\mu$ (i.e. integral over all of X) and then defined $\int_E f\,d\mu$ as $\int_X f\cdot\chi_E\,d\mu$. This is a matter of taste. Note that (b) follows from (a) and (f) since $f\cdot\chi_A \leq f\cdot\chi_B$.

If $\mathcal{M}_E = \{A \in \mathcal{M} \mid A \subseteq E\}$, then $\mathcal{M}_E$ is a σ-algebra on E (Since $E \in \mathcal{M}$, $E \in \mathcal{M}_E$). If

$$\mu_E := \mu|_{\mathcal{M}_E}$$

then μ_E is a positive measure on $(E, \mathcal{M}_E)$. One checks easily from the definitions that

$$\int_E f\,d\mu = \int_E (f|_E)\,d\mu_E.$$

Proposition: Let s, t be non-negative m'ble simple functions on X. For $E \in \mathcal{M}$, define

$$\varphi(E) = \int_E s\,d\mu.$$

Then φ is a measure on $\mathcal{M}$. Also

$$\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu.$$

The above is a provisional form of more general theorems.

Proof:

If $E = \bigsqcup_{r=1}^{\infty} E_r$ ($\sqcup$ = "disjoint union") with $E_i \in \mathfrak{M}$, $i \in \mathbb{N}$, then (with $s = \sum_1^n \alpha_i \chi_{A_i}$ where $\alpha_1, \dots, \alpha_n$ are the distinct values of s)

$$\varphi(E) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$$

$$= \sum_{i=1}^{n} \sum_{r=1}^{\infty} \alpha_i \mu(A_i \cap E_r)$$

$$= \sum_{r=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E_r)$$

$$= \sum_{r=1}^{\infty} \int_{E_r} s \, d\mu$$

$$= \sum_{r=1}^{\infty} \varphi(E_r)$$

Further $\varphi(\phi) = 0$, whence φ is not identically ∞. Thus φ is a positive measure.

Next suppose $t = \sum_{i=1}^{m} \beta_i \chi_{B_i}$, $\beta_1, \dots, \beta_m$ the distinct values of t. Note $B_j = s^{-1}(\beta_j)$, $j = 1, \dots, m$.

Set $E_{ij} = A_i \cap B_j$

Note X is the disjoint union of the E_{ij}.

It is straightforward to see that

$$\int_{E_{ij}} (s+t) \, d\mu = (\alpha_i + \beta_j) \mu(E_{ij})$$

and

$$\int_{E_{ij}} s \, d\mu + \int_{E_{ij}} t \, d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}).$$

Since X is the disjoint union of the E_{ij} the assertion follows, i.e.

$$\int_X (s+t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu.$$

Theorem (Lebesgue's monotone convergence theorem): Let $\{f_n\}$ be a sequence of m'ble funtions on X and suppose that

(a) $0 \le f_1(x) \le f_2(x) \le \dots \le \infty \quad \forall x \in X$,

(b) $f_n(x) \longrightarrow f(x)$ as $n \to \infty \quad \forall x \in X$.

Then f is m'ble and

$$\int_X f_n \, d\mu \longrightarrow \int_X f \, d\mu \quad \text{as } n \longrightarrow \infty$$

Proof:

Since $\{\int_X f_n \, d\mu\}$ is a non-decreasing seq in $[0,\infty]$ therefore $\exists\, \alpha \in [0,\infty]$ s.t.

$$\int_X f_n \, d\mu \longrightarrow \alpha \quad \text{as} \quad n \longrightarrow \infty.$$

Since $\int_X f_n \, d\mu \le \int_X f \, d\mu \quad \forall n$ therefore

$$\alpha \le \int_X f \, d\mu.$$

We have to show $\alpha \geq \int_X f \, d\mu$.

Let s be a simple m'ble funtion such that

$$0 \le s \le f.$$

Fix a number c in the interval $(0,1)$, so that $0 < c < 1$

For $n \in \mathbb{N}$ define

$$E_n = \{x \in X \mid f_n(x) \geq c\, s(x)\}$$

Then each E_n is m'ble (if $g_n = f - cs$, then $E_n = g_n^{-1}[0,\infty]$) and

$$E_1 \subset E_2 \subset \dots \subset E_n \subset E_n \subset \dots$$

and we have

$$X = \bigcup_{n=1}^{\infty} E_n.$$

To see these two relations, since $f_n \le f_{n+1}$ it follows that $f_n(x) \geq cs(x) \Rightarrow f_{n+1}(x) \geq s(x)$, i.e. $E_n \subset E_{n+1}$. Next, suppose $x \in X$. Since $0 < c < 1$, therefore $cs(x) < f(x)$. Since $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$, $\exists\, n \in \mathbb{N}$ such that $f_n(x) \geq cs(x)$. This means $x \in E_n$ for this n, and hence $\bigcup_n E_n = X$.

We have shown that

$$E \longmapsto \int_E cs \, d\mu \quad , \; E \in \mathfrak{M}$$

is a measure on $\mathfrak{M}$ since cs is simple, m'ble and non-negative. It follows that

$$\lim_{n \to \infty} \int_{E_n} cs \, d\mu = \int_X cs \, d\mu = c \int_X s \, d\mu .$$

Now $\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} cs \, d\mu$

Letting $n \to \infty$, we get

$$\alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \geq c \int_X s \, d\mu \quad \text{———} \quad (*)$$

Now $(*)$ holds for every $c \in (0,1)$ and hence

$$\alpha \geq \int_X s \, d\mu .$$

Since s is an arbitrary simple m'ble function s.t. $0 \le s \le f$, taking supremums we get

$$\alpha \geq \int_X f \, d\mu.$$

This proves the result.

Theorem: If $f_n : X \longrightarrow [0,\infty]$ is m'ble for $n \in \mathbb{N}$ and
$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in X),$$
then
$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof:

Let $\{s_n\}$ and $\{t_n\}$ be simple m'ble functions s.t. $0 \le s_n, t_n$ and $s_n \uparrow f_1$ and $t_n \uparrow f_2$. We know such sequences exist. Further, we have shown that $\int_X (s_n + t_n) d\mu = \int_X s_n \, d\mu + \int_X t_n \, d\mu$.
Since $0 \le s_1 + t_1 \le s_2 + t_2 \le \dots$ and $s_n + t_n \longrightarrow f_1 + f_2$, from the monotone convergence theorem we see that
$$\int_X (f_1 + f_2) d\mu = \int_X f_1 \, d\mu + \int_X f_2 \, d\mu.$$
By induction if $g_N = f_1 + f_2 + \dots + f_N$, then
$$\int_X g_N \, d\mu = \sum_{n=1}^{N} \int_X f_n \, d\mu$$

Now $g_N \longrightarrow f$ and $0 \le g_1 \le g_2 \le \dots$. Hence by the MCT (the "monotone convergence theorem")
$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$
as required.

q.e.d.

Remarks

1. Recall that a sequence $\{a_n\}$ can be regarded as a function on $\mathbb{N}$. If $f = \{a_n\}$ is a sequence with $a_n \in [0,\infty]$, $n \in \mathbb{N}$,

and μ is the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} a_n.$$

2. The above theorem translates to a well-known theorem about double series of non-negative extended numbers, namely:

Corollary: If $a_{ij} \geq 0$ for $i, j \in \mathbb{N}$, then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Theorem (Fatou's lemma): If $f_n : X \longrightarrow [0, \infty]$ is m'ble for each $n \in \mathbb{N}$, then

$$\int_X \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.$$

Proof:

For $k \in \mathbb{N}$ define

$$g_k = \inf_{n \geq k} f_n.$$

Let

$$g = \liminf_{n \to \infty} f_n$$

Then $g_k : X \longrightarrow [0, \infty]$ is m'ble. Moreover

(a) $0 \leq g_1 \leq g_2 \leq \dots$ and $g_k \longrightarrow g$ as $k \to \infty$

(b) $g_k \leq f_n \quad \forall \; n \geq k, \; k \geq 1$

From (b) we see that for $k \geq 1$,

$$\int_X g_k \, d\mu \le \int_X f_n \, d\mu \quad \forall\ n \geq k,$$

whence $$\int_X g_k \, d\mu \le \inf_{n \geq k} \int_X f_n \, d\mu.$$

The right side is an increasing sequence in k whose limit as $k \to \infty$ is by definition $\liminf_{n\to\infty} \int_X f_n \, d\mu$. Thus letting $k \to \infty$, and using the MCT for $\{g_k\}$ we get

$$\int_X g \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

q.e.d.

Theorem: Suppose $f: X \longrightarrow [0,\infty]$ is m'ble, and

$$\varphi(E) = \int_E f \, d\mu \qquad (E \in \mathcal{M}).$$

Then φ is a measure on $\mathcal{M}$, and

$$\int_X g \, d\varphi = \int_X gf \, d\mu$$

for every m'ble g on X with range $[0,\infty]$.

Proof:

Let $E_1, E_2, E_3, \ldots$ be a sequence of disjoint members of $\mathcal{M}$, and let $E = \bigcup_n E_n$.

Then

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f.$$

From an earlier theorem, this means

$$\int_X \chi_E f \, d\mu = \sum_{j=1}^{\infty} \int_X \chi_{E_j} f \, d\mu$$

whence

$$\int_E f\,d\mu = \sum_{j=1}^{\infty} \int_{E_j} f\,d\mu.$$

This means $\qquad \phi(E) = \sum_{j=1}^{\infty} \phi(E_j).$

Moreover, clearly $\phi(\emptyset) = 0$. Hence ϕ is a positive measure on $\mathcal{M}$.

Suppose s is a simple m'ble function on X, taking values in $[0,\infty)$. Writing

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

with $\alpha_1, \dots, \alpha_n$ distinct non-negative real numbers, and A_i m'ble, we see from earlier results that

$$\int_X s\,d\phi = \sum_{i=1}^{n} \alpha_i \phi(A_i)$$

$$= \sum_{i=1}^{n} \alpha_i \int_{A_i} f\,d\mu$$

$$= \sum_{i=1}^{n} \alpha_i \int_X \chi_{A_i} f\,d\mu$$

$$= \int_X \left(\sum_{i=1}^{n} \alpha_i \chi_{A_i}\right) f\,d\mu$$

$$= \int_X s f\,d\mu.$$

Now suppose g is as in the statement of the theorem. Pick a sequence of m'ble simple functions $\{s_n\}$ with $0 \le s_1 \le s_2 \le \dots$, $s_n \to g$ as $n \to \infty$. Then we have by MCT (applied more than once)

$$\int_X g\,d\varphi = \lim_{n\to\infty} \int_X s_n\,d\varphi$$

$$= \lim_{n\to\infty} \int_X s_n f\,d\mu$$

$$= \int_X gf\,d\mu .$$

The last equality is by the MCT applied to the sequence $\{s_n f\}$. Note that $0 \le s_1 f \le s_2 f \le \ldots$ and $\lim_{n\to\infty} s_n f = gf$.

q.e.d.