August 16, 2018 Lecture 3

Simple functions

Let X be a measurable space.

Definition: A function $s: X \longrightarrow \mathbb{C}$ is said to be *simple* if its range $s(X)$ is a finite set.

Suppose s is simple and $s(X) = \{\alpha_1, \ldots, \alpha_n\}$ with α_i's distinct. Let $A_i = s^{-1}(\alpha_i)$, $i = 1, \ldots, n$. Then $A_1, \ldots, A_n$ are pairwise disjoint, i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$.

Note that

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

Conversely, any function s which can be represented as above must take only a finite number of values and is therefore simple.

Theorem: Let $f: X \longrightarrow [0, \infty]$ be a m'ble function. There exist simple m'ble functions s_n, $n \in \mathbb{N}$, such that

(a) $0 \leq s_1 \leq s_2 \leq \ldots \leq f$

(b) $s_n(x) \longrightarrow f(x)$ as $n \to \infty$ for every $x \in X$.

Proof:

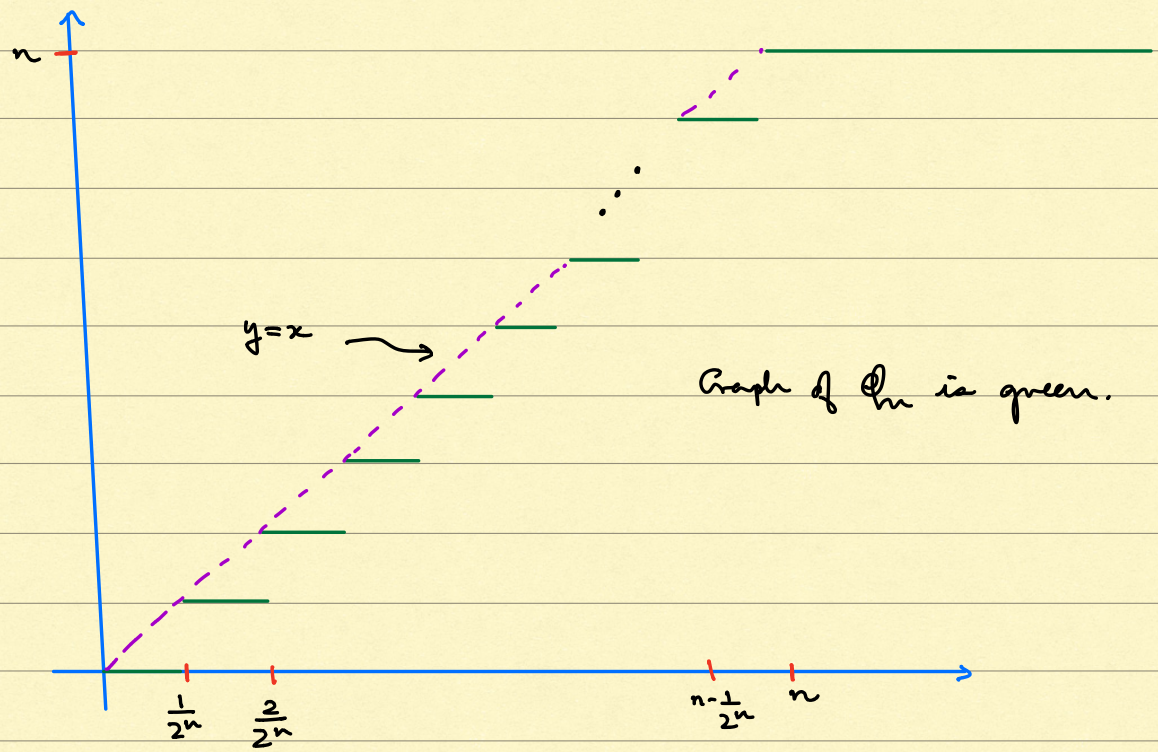
For each $n \in \mathbb{N}$, define

$$\varphi_n : [0, \infty] \longrightarrow [0, \infty)$$

note this! ∞ is included.

as follows:

$$\varphi_n(t) = \begin{cases} \dfrac{i}{2^n} & \text{if } t < n \ \underline{\underline{\text{and}}} \ \dfrac{i}{2^n} \le t < \dfrac{i+1}{2^n} \\ n & \text{if } n \le t \le \infty. \end{cases}$$



Graph of φ_n is green.

Now clearly

$$t - \frac{1}{2^n} < \varphi_n \le t \qquad 0 < t \le n$$

and

$$\left.\begin{array}{l} 0 \le \varphi_1(t) \le \varphi_2(t) \le \ldots \le t \\ \text{and} \quad \varphi_n(t) \longrightarrow t \ \text{ as } n \to \infty \end{array}\right\} \ t \in [0,\infty].$$

Set $s_n = \varphi_n \circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "s_n"'] & {[0,\infty]} \arrow[d, "\varphi_n"] \\ & {[0,\infty)} \end{tikzcd} \quad \varphi_n \leftarrow \text{simple.}$$

φ_n is m'ble & simple. This means s_n is m'ble and simple. Clearly s_n satisfies conditions (a) and (b) of the theorem.

Measures

Definition: Let $(X, \mathcal{M})$ be a m'ble space. A positive measure μ on $\mathcal{M}$ (or on X, if $\mathcal{M}$ is understood from the context) is a map

$$\mu : \mathcal{M} \longrightarrow [0, \infty]$$

such that if $\{A_n\}$ is a disjoint countable collection of members of $\mathcal{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \qquad \text{———} (*)$$

To avoid annoying trivialities, we always assume there is an $A \in \mathcal{M}$ such that $\mu(A) < \infty$.

A measure space is a triple $(X, \mathcal{M}, \mu)$ where $(X, \mathcal{M})$ is a measurable space and μ is a positive measure on $\mathcal{M}$.

A complex measure μ on a σ-algebra $\mathcal{M}$ is a map $\mu : \mathcal{M} \longrightarrow \mathbb{C}$ such that $(*)$ holds for any countable disjoint collection $\{A_n\}$ of members of $\mathcal{M}$.

Remarks: 1. For any non-empty subset $\mathcal{C}$ on $\mathcal{P}(X)$, a map μ on $\mathcal{C}$ (to any subset of $[-\infty, \infty]$ or $\mathbb{C}$) is called countably additive if $(*)$ holds for any countable disjoint sequence $\{A_n\}$ in $\mathcal{C}$.

2. Real measures are a subset of complex measures.

3. Positive measures are really non-negative measures which can take ∞ as a value, and s.t. $\mu(A) < \infty$ for some $A \in \mathcal{M}$.

Theorem: Let μ be a positive measure on $(X, \mathcal{M})$.

(a) $\mu(\phi) = 0$.

(b) If $A_1, \ldots, A_n$ are pairwise disjoint m'ble sets then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

(c) $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$, $A, B \in \mathcal{M}$

(d) If $A_n \in \mathcal{M}$, $n \in \mathbb{N}$, with $A_1 \subset A_2 \subset \ldots \subset A_n \subset A_{n+1} \subset \ldots$

and $A = \bigcup_{n=1}^{\infty} A_n$, then

$$\mu(A) = \lim_{n \to \infty} \mu(A_n), \quad \mathcal{M}$$

(e) If $A_n \in \mathcal{M}$, $n \in \mathbb{N}$, with $A_1 \supset A_2 \supset \ldots \supset A_n \supset A_{n+1} \supset \ldots$,

and $A = \bigcap_{n=1}^{\infty} A_n$, and if $\mu(A_1) < \infty$, then

$$\mu(A) = \lim_{n \to \infty} \mu(A_n)$$

Proof:

(a) Let $A \in \mathcal{M}$ be s.t. $\mu(A) < \infty$. Let $A_1 = A$ and $A_n = \phi$ for $n \geq 2$. Apply countable additivity to get

$$\mu(A) = \mu(A) + \sum_{n=2}^{\infty} \mu(\phi)$$

It follows that $\mu(\phi) = 0$

(b) Take $A_{n+k} = \phi$ for $k \geq 1$ apply countable additivity to $A_1, A_2, \ldots, A_n, \underbrace{A_{n+1}, A_{n+2}, \ldots}_{\text{all empty sets}}$ and use part (a).

(c) Since $B = A \sqcup (B \setminus A)$ where $\sqcup$ denotes disjoint union, therefore $\mu(B) = \mu(A) + \mu(B \setminus A)$ and this proves $\mu(B) \geq \mu(A)$

(d) Let $B_1 = A_1$, and for $n \geq 2$, set $B_n = A_n \setminus A_{n-1}$. Then $\{B_n\}$ is a sequence of pairwise disjoint sets. Note that $A = \bigcup_{n=1}^{\infty} B_n$. Thus by countable additivity

$$\mu(A) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) \qquad \text{—— (1)}$$

Now $B_1 \cup B_2 \cup \dots \cup B_n = A_n$, and hence by (b) $\sum_{k=1}^{n} \mu(B_k) = \mu(A_n)$. Substituting in (1) we get the answer.

(e) Let $B_n = A_1 \sim A_n$. Then

$$\phi = B_1 \subset B_2 \subset \dots \subset B_n \subset B_{n+1} \subset \dots$$

and hence by (c)

$$\mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \lim_{n\to\infty} \mu(B_n) \qquad \text{—— (2)}$$

Now $\bigcup_{n=1}^{\infty} B_n = A_1 \sim \bigcap_{n=1}^{\infty} A_n$ and hence

$$\mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \mu(A_1) - \mu\Big(\bigcap_{n=1}^{\infty} A_n\Big).$$

Thus by (*)

$$\mu(A_1) - \mu\Big(\bigcap_{n\geq 1} A_n\Big) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \big(\mu(A_1) - \mu(A_n)\big)$$
$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_n).$$

Since $\mu(A_1) < \infty$, the result follows.

q.e.d.

The so-called counting measure on $\mathbb{N}$. See below for more on the counting measure

Example : On $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ consider the measure $\mu(A) = \infty$ if A is infinite & $\mu(A) =$ cardinality of A otherwise. Let $A_n = \{n, n+1, n+2, \dots\}$. Then $A_1 \supset A_2 \supset \dots$. $\bigcap_n A_n = \phi$. It is therefore clear that $\mu(\bigcap_n A_n) \neq \lim_{n\to\infty} \mu(A_n)$. The problem is that $\mu(A_1) = \infty$.

Examples of measures:

Interesting and natural measures are not straightforward to construct. The important Lebesgue measure on $\mathbb{R}$ requires some work to construct. Here are a few simple minded examples.

(a) Let X be a set. On $(X, \mathcal{P}(X))$ define a measure μ by $\mu(A) = \infty$ if A is infinite and $\mu(A)$ is the cardinality of A if A is finite. It is easy to see this is a measure. This is called the counting measure on X.

(b) Fix $x_0 \in X$. Let $E \in \mathcal{P}(X)$. Define

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

This measure is called the unit mass concentrated at x_0 or more commonly the Dirac measure at x_0. The σ-algebra need not be $\mathcal{P}(X)$. Any σ-algebra on X will do. The measure is often denoted δ_{x_0}.