LECTURE 24

Date of Lecture: November 10, 2018

All vector spaces are over $\mathbb{C}$. As before m is the measure on the Lebesgue σ-algebra on $\mathbb{R}$ given by

$$m = \frac{\text{Lebesgue measure}}{\sqrt{2\pi}}$$

and

$$\int_a^b f(x)\,dx$$

will denote the integral of a measurable function f on $\mathbb{R}^1$ with respect to the Lebesgue measure over (a,b). Similarly $\int_a^b f\,dm = \int_a^b f(x)\,dm(x) = \int_{(a,b)} f\,dm$. Moreover,

$$L^p := L^p(m) \qquad (1 \le p \le \infty),$$

and for a measurable function f on $\mathbb{R}$

$$\|f\|_p := \left\{\int_{-\infty}^{\infty} |f|^p\,dm\right\}^{\frac{1}{p}}.$$

1. Fourier Inversion

1.1. Recap from last lecture.

Last lecture (Lecture 23 on Nov 6) we proved the following two results.

Theorem 1.1.1. *For any function f on $\mathbb{R}$, and any real number y, let $f_y(x) = f(x-y)$. If $f \in L^p$, then the mapping*

$$y \mapsto f_y$$

is a uniformly continuous map from $\mathbb{R}$ to L^p.

Theorem 1.1.2. [the Riemann-Lebesgue Theorem]*Let $f \in L^1$. Then $\widehat{f} \in C_0(\mathbb{R})$ and $\|\widehat{f}\|_\infty \le \|f\|_1$.*

1.2. The functions H_λ and h_λ.

For $\lambda > 0$ let

$$H_\lambda(t) = e^{-\lambda|t|} \qquad (t \in \mathbb{R}) \tag{1.2.1}$$

and

$$h_\lambda(x) = \int_{-\infty}^{\infty} H_\lambda(t)e^{itx}\,dm(t) \qquad (x \in \mathbb{R}). \tag{1.2.2}$$

It is easy to see that

$$h_\lambda(x) = \sqrt{\frac{2}{\pi}}\frac{\lambda}{\lambda^2+x^2} \tag{1.2.3}$$

[1]Which is either taking values in $[0,\infty]$ or is integrable with respect to the Lebesgue measure

whence,

$$\int_{-\infty}^{\infty} h_\lambda(x)\,dm(x) = 1. \tag{1.2.4}$$

Proposition 1.2.5. *If $f \in L^1$, then*

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H_\lambda(t)\widehat{f}(t)e^{itx}\,dm(t) \qquad (x \in \mathbb{R}).$$

Proof.

$$\begin{aligned}
(f * h_\lambda)(x) &= \int_{-\infty}^{\infty} f(x-y)h_\lambda(y)\,dm(y)\\
&= \int_{-\infty}^{\infty} f(x-y)\left\{\int_{-\infty}^{\infty} H_\lambda(t)e^{ity}\,dm(t)\right\}dm(y)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)H_\lambda(t)e^{it(y-x)}e^{itx}\,dm(y)\,dm(t) \qquad \text{(Fubini)}\\
&= \int_{-\infty}^{\infty} H_\lambda(t)e^{itx}\left\{\int_{-\infty}^{\infty} f(x-y)e^{-it(x-y)}\,dm(y)\right\}dm(t)\\
&= \int_{-\infty}^{\infty} H_\lambda(t)\widehat{f}(t)e^{itx}\,dm(t).
\end{aligned}$$

□

Theorem 1.2.6. *If $g \in L^\infty$ and g is continuous at a point x, then*

$$\lim_{\lambda\to 0}(g * h_\lambda)(x) = g(x).$$

Proof.

$$\begin{aligned}
(g * h_\lambda)(x) - g(x) &= \int_{-\infty}^{\infty} (g(x-y) - g(x))h_\lambda(y)\,dm(y)\\
&= \int_{-\infty}^{\infty} (g(x-y) - g(x))\frac{1}{\lambda}h_1\left(\frac{y}{\lambda}\right)dm(y)\\
&= \int_{-\infty}^{\infty} (g(x-\lambda u) - g(x))h_1(u)\,dm(u).
\end{aligned}$$

The absolute value of the last integrand is dominated by $2\|g\|_\infty h_1(u)$, and this is in L^1, which means DCT applies and we can take the limit through the integral sign as λ tends to 0. The result follows immediately. □

Theorem 1.2.7. *If $p \in [1, \infty)$ and $f \in L^p$, then*

$$\lim_{\lambda\to 0}\|f * h_\lambda - f\|_p = 0.$$

Proof. We claim that $h_\lambda \in L^q$ for every $q \in [1, \infty]$. First note that if

$$E = \{x \mid h_\lambda(x) > 1\},$$

then as $h_\lambda \in L^1$, $m(E) < \infty$. Let $F = \{x \mid h_\lambda(x) \le 1\}$. Now $\|h_\lambda\|_\infty = 1/\lambda < \infty$, so that $h_\lambda \in L^\infty$. For $1 \le q < \infty$ we have $h_\lambda^q(x) \le h_\lambda(x)$ for $x \in F$, whence

$$\begin{aligned}\int_{-\infty}^{\infty} h_\lambda^q(x)\,dm(x) &= \int_F h_\lambda^q\,dm + \int_E h_\lambda^q\,dm \\ &\le \int_F h_\lambda\,dm + \|h_\lambda\|_\infty^q\, m(E) < \infty,\end{aligned}$$

proving the claim. This means $f * h_\lambda(x)$ is defined everywhere.[2] Now,

$$(f * h_\lambda)(x) - f(x) = \int_{-\infty}^{\infty} (f(x-y) - f(x)) h_\lambda(y)\,dm(y).$$

Since $h_\lambda\,dm$ gives a probability measure and since $s \mapsto s^p$ is convex, by Jensen's inequality we get

$$(**) \qquad \Big|(f * h_\lambda)(x) - f(x)\Big|^p \le \int_{-\infty}^{\infty} \Big|f(x-y) - f(x)\Big|^p h_\lambda(y)\,dm(y).$$

Integrating both sides and applying Fubini we get

$$\|f * h_\lambda - f\|_p^p \le \int_{-\infty}^{\infty} \|f_y - f\|_p^p h_\lambda(y)\,dm(y).$$

If g is the function $g(y) = \|f_{-y} - f\|_p^p$, then by Theorem 1.1.1, g is continuous on $\mathbb{R}$. Moreover, the integral on the right side of $(**)$ becomes $g * h_\lambda(0)$, whence by Theorem 1.2.6, the right side of $(**)$ converges to $g(0) = 0$ as $\lambda \to 0$. □

Theorem 1.2.8. [The Fourier Inversion Theorem] *Suppose f and $\widehat{f}$ are in L^1, and*

$$g(x) = \int_{-\infty}^{\infty} \widehat{f}(t) e^{itx}\,dm(t) \qquad (x \in \mathbb{R}).$$

Then $g \in C_0(\mathbb{R})$ and $f(x) = g(x)$ a.e.

Proof. By Proposition 1.2.5

$$(\dagger) \qquad (f * h_\lambda)(x) = \int_{-\infty}^{\infty} H_\lambda(t)\widehat{f}(t) e^{itx}\,dm(t)$$

for $x \in \mathbb{R}$. Since

$$\Big|H_\lambda(t)\widehat{f}(t)e^{itx}\Big| \le |\widehat{f}(t),$$

and since by our hypothesis $\widehat{f} \in L^1$, DCT applies. Now $\lim_{\lambda\to 0} H_\lambda(t) = 1$ for every $t \in \mathbb{R}$ as is easily checked from (1.2.1). So applying DCT, and Theorem 1.2.6 we get that the right side of $(\dagger)$ converges to $g(x)$ for every x as $\lambda \to 0$. On the other hand, by Theorem 1.2.7 and the fact that if $\{P_n\}$ is a sequence in L^p converging in L^p to P, then there is a subsequence $\{P_{n_k}\}$ which converges pointwise almost everywhere to P (see [Lecture 12, p.5]) we see that there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ and

$$(\ddagger) \qquad \lim_{n\to\infty} (f * h_\lambda)(x) = f(x) \text{ a.e.}$$

[2] Let q be the exponent conjugate to p. Let $\phi(y) = f(-y)$. Then $\phi \in \text{Ł}^p$, and hence so does ϕ_x. Since $(f * h_\lambda(x) = \int_{-\infty}^{\infty} \phi_x(y) h_\lambda(y) dm(y)$, and $h_\lambda \in L^q$, we see that $(f * h_\lambda)(x)$ is defined for every x.

We have shown that the right side (†) converges to $g(x)$, which means by $f(x) = g(x)$ for almost every x by (‡). Finally, by definition, $g = \widehat{\psi}$ where $\psi(x) = \widehat{f}(-x)$. It follows that $g \in C_0(\mathbb{R})$ by the Riemann-Lebesgue Theorem, i.e. Theorem 1.1.2. □

Corollary 1.2.9. [The Uniqueness Theorem] *If $f \in L^1$ and $\widehat{f}(t) = 0$ for all $t \in \mathbb{R}$, then $f(x) = 0$ a.e.*

Proof. Since $\widehat{f} = 0$, therefore $\widehat{f} \in L^1$ and Theorem 1.2.8 applies. □

2. The Plancheral Theorem

2.1. **Fourier transform on $L^1 \cap L^2$.** Suppose $f \in L^1 \cap L^2$. We will show that

$$\widehat{f} \in L^2 \tag{2.1.1}$$

and that

$$\|f\|_2 = \|\widehat{f}\|_2. \tag{2.1.2}$$

Let

$$\widetilde{f}(x) = \overline{f(-x)}$$

for $x \in \mathbb{R}$ and

$$g = f * \widetilde{f}. \tag{2.1.3}$$

We also know from the previous lecture that $\widehat{\widetilde{f}} = \overline{\widehat{f}}$. and hence

$$\widehat{g} = |\widehat{f}|^2 \tag{2.1.4}$$

Now

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x-y)\overline{f(-y)}\, dm(y) \\ &= \int_{-\infty}^{\infty} f(y+x)\overline{f(y)}\, dm(y) \\ &= \int_{-\infty}^{\infty} f_{-x}(y)\overline{f(y)}\, dm(y), \end{aligned}$$

giving

$$g(x) = \langle f_{-x}, f\rangle \qquad (x \in \mathbb{R}). \tag{2.1.5}$$

We know from Theorem 1.1.1 that $x \mapsto f_{-x}$ is a continuous map from $\mathbb{R}$ to L^2, and we know that $h \mapsto \langle h, f\rangle$ is a continuous map from L^2 to $\mathbb{C}$. It follows from (2.1.5) that g is continuous. Moreover, applying Cauchy-Schwarz to (2.1.5), we see that

$$|g(x)|^2 \le \|f\|_2^2,$$

whence g is bounded. Theorem 1.2.6 therefore applies to g for every $x \in \mathbb{R}$ and we get

$$\begin{aligned} \|f\|_2^2 = g(0) &= \lim_{\lambda\to 0}(g * h_\lambda)(0) && \text{(Theorem 1.2.6)} \\ &= \lim_{\lambda\to 0}\int_{-\infty}^{\infty} H_\lambda(t)\widehat{g}(t)\, dm(t) && \text{(Theorem 1.2.5).} \end{aligned} \tag{2.1.6}$$

On the other hand, by (2.1.4) we have

$$H_\lambda(t)\widehat{g}(t) = H_\lambda(t)|\widehat{f}(t)|^2$$

which is a positive and increases to $\widehat{f}(t)|^2$ as $\lambda \to 0$, so that MCT applies and we have

$$\lim_{\lambda\to 0}\int_{-\infty}^{\infty} H_\lambda(t)\widehat{g}(t)\,dm(t) = \int_{-\infty}^{\infty} |\widehat{f}(t)|^2\,dm(t) = \|\widehat{f}\|_2^2. \tag{2.1.7}$$

Thus (2.1.6) and (2.1.7) establish (2.1.2) for $f \in L^1 \cap L^2$. This also shows that $\widehat{f} \in L^2$ for every $f \in L^1 \cap L^2$.

Next, we wish to extend the Fourier transform from $L^1 \cap L^2$ to L^2 in a surjective norm-preserving way. More precisely let $Y \subset L^2$ be the collection

$$Y = \{g \in L^2 \mid g = \widehat{f}, \text{ for some } f \in L^1 \cap L^2\}.$$

Let

$$\Phi\colon L^1 \cap L^2 \to Y$$

be the map

$$\Phi(f) = \widehat{f}.$$

Φ is a surjective isometry. Let X be the closure of Y in L^2. Now, $L^1 \cap L^2$ is dense in L^2. Since Φ is a continuous linear transformation, it is uniformly continuous on $L^1 \cap L^2$ and hence extends to a map (necessarily a linear transformation as is easy to check via limits)

$$\Phi\colon L^2 \to X$$

and according to [Lecture 17, p.4, Lemma 1.5.1], Φ is a surjective isometry from L^2 to X. We claim that $X = L^2$. This is equivalent to claiming that Y is dense in L^2. Since L^2 is a Hilbert space, this amounts to showing that if $w \in Y^\perp$ then $w = 0$. Consider the collection of functions $\{\varphi_{\lambda,\alpha} \mid \alpha \in \mathbb{R}, \lambda > 0\}$, where $\varphi_{\lambda,\alpha}(x) = e^{i\alpha x} H_\lambda(x)$. Then $\varphi_{\lambda,\alpha} \in L^1 \cap L^2$ and $\widehat{\varphi}_{\lambda,\alpha}(t) = h_\lambda(\alpha - t)$, which means $t \mapsto h_\lambda(\alpha - t)$ is in Y. Now suppose $w \in Y^\perp \subset L^2$. then

$$h_\lambda * \overline{w}(\alpha) = \int_{-\infty}^{\infty} h_\lambda(\alpha - t)\overline{w}(t)\,dm(t) = 0.$$

Letting $\lambda \to 0$ and using Theorem 1.2.6 we get that $\overline{w}(\alpha) = 0$ for every $\alpha \in \mathbb{R}$, which means $w = 0$. Thus Y is dense in L^2, and we therefore have a surjective isometry

$$\Phi\colon L^2 \xrightarrow{\sim} L^2$$

which on $L^1 \cap L^2$ sends f to $\widehat{f}$.

We have now proven parts (a), (b), and (c) of the following theorem (see [R, p. 186, Thm. 9.1.3])

Theorem 2.1.8. *One can associate to each $f \in L^2$ a function $\widehat{f} \in L^2$ so that the following properties hold:*

(a) *If $f \in L^1 \cap L^2$ then $\widehat{f}$ is the previously defined Fourier transform of f.*
(b) *For evry $f \in L^2$, $\|\widehat{f}\|_2 = \|f\|_2$.*
(c) *The mapping $f \mapsto \widehat{f}$ is a Hilbert space isomorphism of L^2 onto L^2.*
(d) *The following symmetric relation exists between f and $\widehat{f}$: If*

$$\varphi_A(t) = \int_{-A}^{A} f(x)e^{-ixt}\,dm(x) \text{ and } \psi_A(x) = \int_{-A}^{A} \widehat{f}(t)e^{ixt}\,dm(t),$$

then $\|\varphi_A - \widehat{f}\|_2 \longrightarrow 0$ and $\|\psi_A - f\|_2 \longrightarrow 0$ as $A \longrightarrow \infty$.

Proof. We only have to prove (b) having already proved the other parts. Now $\|f\chi_{[-A,A]} - f\|_2 \longrightarrow 0$ as $A \longrightarrow \infty$. By definition, $\varphi_A = (f\chi_{[-A,A]})^\wedge$ and hence

$$\|\widehat{f} - \varphi_A\|_2 = \|(f - f\chi_{[-A,A]})^\wedge\|_2 = \|f - f\chi_{[-A,A]}\|_2 \longrightarrow 0$$

as $A \to \infty$.

The same proof works for the other half of (d). $\square$/

References

[R] W. Rudin, *Real and Complex Analysis*, (Third Edition), McGraw-Hill, New York, 1987.