Nov 8, 2018

Lecture 23

Last time we proved the following two lemmas:

Lemma A: Let $(X, \mathcal{M}, \mu)$ be a measure space and $(X, \hat{\mathcal{M}}, \hat{\mu})$ be its completion. If f is $\hat{\mathcal{M}}$-m'ble then $\exists$ an $\mathcal{M}$-m'ble function g such that $f = g$ a.e. $[\hat{\mu}]$.

Lemma B: Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete, σ-finite measure spaces. Let $(X \times Y, \mathcal{M}, \lambda)$ be the completion of $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times \nu)$. If h is an $\mathcal{M}$-m'ble function on $X \times Y$ and $h = 0$ a.e. $[\lambda]$, then $\exists\ N \in \mathcal{S}$, $\mu(N) = 0$ such that for each $x \in X - N$ there exists $F(x) \in \mathcal{T}$ with the property that $\mu(F(x))$ and $h_x(y) = 0$ for all $y \in Y - F(x)$.

Products of complete measures

In what follows $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are complete σ-finite measure spaces, and $(\mathcal{M}, \lambda)$ is the completion of $(\mathcal{S} \times \mathcal{T}, \mu \times \nu)$.

Theorem (Tonelli for complete measures): Let $f : X \times Y \to [0, \infty]$ be $\mathcal{M}$-m'ble. Then f_x (resp. f^y) is $\mathcal{T}$-m'ble (resp. $\mathcal{S}$-m'ble) for almost every $x \in X$ (resp. almost every $y \in Y$). Let φ_f and ψ_f be be the functions defined almost everywhere on X and Y respectively by

$$\varphi_f(x) = \int_Y f_x \, d\nu, \qquad \psi_f(y) = \int_Y f^y \, d\nu. \qquad \text{——— (1)}$$

Then

$$\int_X \varphi_f \, d\mu = \int_{X\times Y} f \, d\lambda = \int_Y \psi_f \, d\nu \qquad \text{——— (2).}$$

Proof: By Lemma A $\exists\, g$ which is $\mathcal{S}\times\mathcal{T}$-m'ble s.t. $f = g$ a.e. $[\lambda]$. Tonelli applies to g. Apply Lemma B to $h = f - g$ to get the result. q.e.d.

An immediate consequence is Fubini for complete measures.

Theorem (Fubini for complete measures): Suppose f is a complex $\mathcal{M}$-m'ble function. Then f_x (resp f^y) is $\mathcal{T}$-m'ble (resp. $\mathcal{S}$-m'ble) for almost every $x \in X$ (resp. almost every $y \in Y$). Moreover

(a) If $\varphi_{|f|}$ is the a.e. defined function $x \mapsto \int_Y |f_x| \, d\nu$, and $\int_X \varphi_{|f|} \, d\mu < \infty$ then $f \in L^1(\lambda)$.

(b) If $f \in L^1(\lambda)$ then for almost all $x \in X$ (resp. almost all $y \in Y$), $f_x \in L^1(\nu)$ (resp. $f^y \in L^1(\mu)$) and if φ_f (resp ψ_f) is the a.e. defined function $x \mapsto \int_Y f_x \, d\nu$ (resp. $y \mapsto \int_X f^y \, d\mu$) then

$$\int_X \varphi_f \, d\mu = \int_{X\times Y} f \, d\lambda = \int_Y \psi_f \, d\nu.$$

Proof: As before, part (a) follows from Tonelli for complete measures. For (b), as before, by breaking f into real and imaginary parts, and then each of those into positive and negative parts, we are reduced to the case where $f \in L^1(\lambda)$ and $f \geq 0$. The result follows from Tonelli for complete measures. q.e.d.

Fourier transforms

From now on

$$m = \frac{\text{Lebesgue measure on } \mathbb{R}}{\sqrt{2\pi}}.$$

Thus

Integration with respect to Lebesgue measure.

$$\int_{-\infty}^{\infty} f(x)\, dm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx. \quad \text{——— (1)}$$

$$\|f\|_p := \left\{ \int_{-\infty}^{\infty} |f(x)|^p\, dm(x) \right\}^{1/p} \qquad (1 \leq p < \infty) \quad \text{—— (2)}$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dm(y) \qquad (x \in \mathbb{R}) \quad \text{—— (3)}$$

and

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)\, e^{-ixt}\, dm(x) \qquad (t \in \mathbb{R}). \quad \text{—— (4)}$$

Of course (1)–(4) are only defined when the right side is sensible.

NOTATION → $\boxed{L^p := L^p(m), \qquad 1 \leq p \leq \infty.}$

Defn: $\hat{f}$ as defined in (4) is called the Fourier transform of f. The map $f \longmapsto \hat{f}$ is also called the Fourier transform.

Theorem: $f, g \in L^1 \Rightarrow f * g \in L^1$, and $\|f*g\|_1 \le \|f\|_1 \cdot \|g\|_1$

Proof: This is essentially Tonelli.

$$|(f*g)(x)| \le \int_{-\infty}^{\infty} |f(x-y)\, g(y)|\, dm(y)$$

$$\Rightarrow \int_{-\infty}^{\infty} |(f*g)(x)|\, dm(x) \le \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x-y)|\, |g(y)|\, dm(y) \right\} dm(x)$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|\, dm(x) \right\} dm(y) \quad \text{(Tonelli)}$$

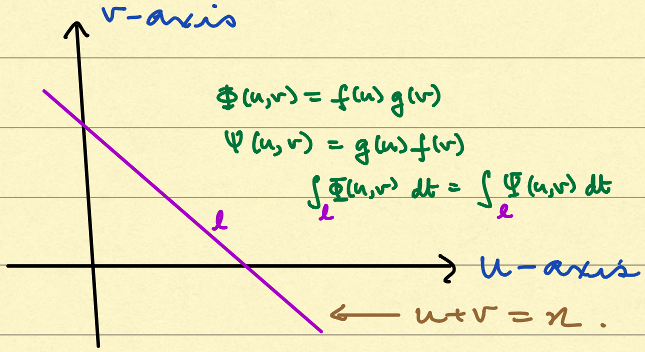
$$= \int_{-\infty}^{\infty} |g(y)| \left\{ \int_{-\infty}^{\infty} |f(x-y)|\, dm(x) \right\} dm(y)$$

$$= \int_{-\infty}^{\infty} |g(y)|\, dm(y) \int_{-\infty}^{\infty} |f(x)|\, dm(x)$$

$$= \|g\|_1 \, \|f\|_1 < \infty$$

q.e.d.

Remark: Standard substitution arguments show that $f*g = g*f$ when $f, g \in L^1$. Indeed setting $s = x-y$ we see that $\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dm(y) = \int_{-\infty}^{\infty} f(s)\, g(x-s)\, dm(s)$, in other words $(f*g)(x) = (g*f)(x)$. We will be using this result without comment.



Intuitive reason why $f*g = g*f$. dt = length element along ℓ. ℓ is $u+v=x$.

Theorem: Suppose $f \in L^1$, $\alpha, \lambda \in \mathbb{R}$.

(a) $g(x) = f(x)e^{i\alpha x} \Rightarrow \hat{g}(t) = \hat{f}(t-\alpha)$

(b) $g(x) = f(x-\alpha) \Rightarrow \hat{g}(t) = \hat{f}(t)\, e^{-i\alpha t}$

(c) $g \in L^1$, $h = f * g \Rightarrow \hat{h}(t) = \hat{f}(t)\, \hat{g}(t)$

(d) $g(x) = \overline{f(-x)} \Rightarrow \hat{g}(t) = \overline{\hat{f}(t)}$

(e) $g(x) = f\left(\frac{x}{\lambda}\right) \Rightarrow \hat{g}(t) = \lambda \hat{f}(\lambda t)$

(f) $g(x) = -ix f(x)$, $g \in L^1 \Rightarrow \hat{f}$ diff'ble and $\hat{f}'(t) = \hat{g}(t)$.

Proof: (a), (b), (d), (e) follow easily from the definitions by direct substitution into the relevant formulae. It remains to to prove (c) and (f).

Here is (c):

$$\hat{h}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itx} f(x-y)\, g(y)\, dm(y)\, dm(x)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itx} f(x-y)\, g(y)\, dm(x)\, dm(y) \qquad \text{(Fubini)}$$

$$= \int_{-\infty}^{\infty} g(y) \left\{ \int_{-\infty}^{\infty} e^{-itx} f(x-y)\, dm(x) \right\} dm(y)$$

$$= \int_{-\infty}^{\infty} g(y)\, e^{-ity} \left\{ \int_{-\infty}^{\infty} e^{-it(x-y)} f(x-y)\, dm(x) \right\} dm(y)$$

$$= \left\{ \int_{-\infty}^{\infty} g(y)\, e^{-ity}\, dm(y) \right\} \left\{ \int_{-\infty}^{\infty} f(u)\, e^{-itu}\, dm(u) \right\}$$

$$= \hat{g}(t)\, \hat{f}(t)$$

giving (c).

Now for (f). Let $\{s_n\}$ be any sequence in $\mathbb{R}$ converging to t. Then

$$\frac{\hat{f}(s_n) - \hat{f}(t)}{s_n - t} = \int_{-\infty}^{\infty} f(x) \left\{ \frac{e^{-is_n x} - e^{-itx}}{s_n - t} \right\} dm(x)$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-itx} \left\{ \frac{e^{-i(s_n - t)x} - 1}{s_n - t} \right\} dm(x).$$

Now $\dfrac{|e^{-i(s_n-t)x} - 1|}{|s_n - t|} \le |x| \quad \forall\ s_n$ and t, and hence DCT applies and letting $n \to \infty$, we get

$$\hat{f}'(t) = \lim_{n \to \infty} \frac{\hat{f}(s_n) - \hat{f}(t)}{s_n - t}$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-itx} \lim_{n \to \infty} \left\{ \frac{e^{-i(s_n - t)x} - 1}{s_n - t} \right\} dm(x)$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-itx} (-ix)\, dm(x)$$

$$= \int_{-\infty}^{\infty} (-ix f(x))\, e^{-itx}\, dm(x)$$

q.e.d.

Towards the Inversion theorem:

Definition: For any function f on $\mathbb{R}$ and every $y \in \mathbb{R}$, f_y is the translate of f defined by

$$f_y(x) = f(x-y), \qquad y \in \mathbb{R}.$$

If y is positive the graph of f_y is the graph of f shifted y units to the right, and if y is negative, the graph shift is $|y|$ units to the left.

Theorem: If $p \in [1,\infty)$ and $f \in L^p$ then the mapping

$$y \longmapsto f_y$$

is a uniformly continuous mapping of $\mathbb{R}$ into L^p, i.e., the map $\gamma_f : \mathbb{R} \longrightarrow L^p$ given by $\gamma_f(y) = f_y$, $y \in \mathbb{R}$, is uniformly continuous.

Proof: (We will temporarily be using the usual Lebesgue measure. It does not affect the proof.)

Let $\varepsilon > 0$ be given. Since $C_c(\mathbb{R})$ is dense in L^p, there exists a cts function g with support $[-A, A]$ such that

$$\|g - f\|_p < \varepsilon.$$

Since g has compact support, it is uniformly continuous. Hence there exists $\delta \in (0, A)$ such that

$$|s - t| < \delta \Rightarrow |g(s) - g(t)| < \frac{\varepsilon}{(3A)^{1/p}}.$$

Note that if $0 < b - a < 2A$ then the support of G given by $G(x) = g(x-b) - g(x-a)$ is contained in $[-A+a, A+a] \cup [-A+b, A+b]$ $= [-A+a, A+b]$, which is an interval of length $2A + (b-a)$.

Now suppose $|s-t| < \delta$. From the above considerations

$$\int_{-\infty}^{\infty} |g(x-s) - g(x-t)|^p dx < \frac{\varepsilon^p}{3A}(2A + \delta) < \varepsilon^p \quad (\text{Since } 0 < \delta < A).$$

Thus

$$\|g_s - g_t\|_p < \varepsilon.$$

Now $\|h_x\|_p = \|h\|_p$ $\forall x$, and all $h \in L^p$, whence

$$\|f_s - f_t\|_p \le \|f_s - g_s\|_p + \|g_s - g_t\|_p + \|g_t - f_t\|_p$$
$$= \|(f-g)_s\|_p + \|g_s - g_t\|_p + \|(f-g)_t\|_p < 3\varepsilon$$

whenever $|s-t| < \delta$. q.e.d.

Theorem: If $f \in L^1$, then $\hat{f} \in C_0$ and

$$\|\hat{f}\|_\infty \le \|f\|_1. \qquad \text{———— } (*)$$

Remark: This is the Riemann-Lebesgue Theorem for Fourier integrals. It is the exact analogue of the Riemann-Lebesgue Theorem for Fourier series.

Proof: The inequality $(*)$ is obvious from the defn of $\hat{f}$.

Suppose $t_n \to t$ as $n \to \infty$. Then

$$\left|\hat{f}(t_n) - \hat{f}(t)\right| \le \int_{-\infty}^{\infty} |f(x)| \cdot \left|e^{-it_n x} - e^{-itx}\right| dm(x).$$

Now $|f(x)| \cdot \left|e^{-itx} - e^{-itx}\right| \le 2|f(x)|$, and $2|f| \in L^1$. Thus DCT applies and we see that $\hat{f}$ is continuous on $\mathbb{R}$.

Next, since $e^{\pi i} = -1$, we get

$$\hat{f}(t) = -\int_{-\infty}^{\infty} f(x)\, e^{-it(x + \pi/t)}\, dm(x)$$

$$= -\int_{-\infty}^{\infty} f\left(x - \frac{\pi}{t}\right) e^{-itx}\, dm(x)$$

$$= -\int_{-\infty}^{\infty} f_{\pi/t}(x)\, e^{-itx}\, dm(x).$$

Thus

$$2\hat{f}(t) = \int_{-\infty}^{\infty} \left\{ f(x) - f_{\pi/t}(x) \right\} e^{-itx}\, dm(x).$$

This gives,

$$2\left|\hat{f}(t)\right| \le \|f - f_{\pi/t}\|_1.$$

The right side tends to 0 as $|t| \to \infty$ by the previous theorem. q.e.d.