Nov 1, 2018

# Lecture 22

## Fubini for complete measures (continued):

In what follows $k, r, s$ are positive integers satisfying

$$\boxed{k = r + s.} \qquad (*)$$

Last time we showed that

$$\mathcal{B}_r \times \mathcal{B}_s = \mathcal{B}_k$$

We also showed (though I forgot to put that in the notes)

**Lemma 1:** Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are two measure spaces and $E \in \mathcal{S}$ is such that $\mu(E) = 0$. Then $\mu \times \nu(E \times F) = 0$ for every $F \in \mathcal{T}$. Similarly if $F \in \mathcal{T}$ and $\nu(F) = 0$, then $\mu \times \nu(E \times F) = 0$ for every $E \in \mathcal{S}$.

*Proof:* Since $\psi_{E\times F} = \mu(E) \cdot \chi_F$ therefore $\psi_{E\times F} = 0$ if $\mu(E) = 0$, whence $(\mu\times\nu)(E\times F) = \int_Y \psi_{E\times F}\, d\nu = 0$. Symmetry proves the second assertion. q.e.d.

**Lemma 2:** On $\mathcal{B}_k = \mathcal{B}_r \times \mathcal{B}_s$ we have $m_k = m_r \times m_s$.

*Proof:* Let $Q_0 = \prod_{i=1}^{k} [0,1) \subset \mathbb{R}^k$. Clearly $m_k(Q_0) = (m_r \times m_s)(Q_0) = 1$. Thus it is enough to show that $m_r \times m_s$ is translation invariant on $\mathcal{B}_k$. We do it by using monotone classes. The argument is by now familiar (or should be), but here it is one more time.

Let $E \in \mathcal{B}_k$. Define

$$E_{\vec{n}} = E \cap (Q_0 + \vec{n}), \qquad \vec{n} \in \mathbb{Z}^k.$$

Then $\{E_{\vec{n}}\}$ is a countable m'ble partition of $E$ s.t. $m_r \times m_s$ and $m_k$ have finite measure on each $E_{\vec{n}}$.

Let $\vec{x} \in \mathbb{R}^k$. Define

$$\mathcal{M} = \{ E \in \mathcal{B}_k \mid m_r \times m_s (E_{\vec{n}}) = m_r \times m_s (E_{\vec{n}} + \vec{x}), \ \forall \vec{n} \in \mathbb{Z}^k \}.$$

If $\vec{x} = (x_1, \dots, x_k)$, set

$$\vec{x}_r = (x_1, \dots, x_r) \in \mathbb{R}^r, \quad \vec{x}_s = (x_{r+1}, \dots, x_k) \in \mathbb{R}^s.$$

If $E = A \times B$, $A \in \mathcal{B}_r$, $B \in \mathcal{B}_s$, then clearly $E + \vec{x} = (A + \vec{x}_r) \times (B + \vec{x}_s)$. Moreover $E_{\vec{n}} = A_{\vec{n}_r} \times B_{\vec{n}_s}$. Thus $A \times B \in \mathcal{M}$ for every $A \in \mathcal{B}_r$, $B \in \mathcal{B}_s$. It follows that $\mathcal{R}(\mathcal{B}_r, \mathcal{B}_s) \subseteq \mathcal{M}$. It is straightforward to see that $\mathcal{M}$ is a monotone class. Since $\sigma(\mathcal{R}) = \mathcal{B}_r \times \mathcal{B}_s = \mathcal{B}_k$ therefore $\mathcal{M} = \mathcal{B}_k$. This gives $m_r \times m_s (E_{\vec{n}}) = m_r \times m_s (E_{\vec{n}} + \vec{x})$, whence $m_r \times m_s (E) = m_r \times m_s (E + \vec{x})$. q.e.d.

Lemma 3: Let $N \in \mathcal{L}_r$ be an $m_r$-null set, i.e. $m_r(N) = 0$. Then $N \times \mathbb{R}^s \in \mathcal{L}_k$ and $m_k(N \times \mathbb{R}^s) = 0$.

Remark: Since we do not know that $\mathcal{L}_r \times \mathcal{L}_s = \mathcal{L}_k$ (in fact it is not true, and the proof is below, and in fact was given in the last class but I forgot to write it) we cannot use Lemma 1 above.

Proof: There exist $B \in \mathcal{B}_r$ such that $m_r(B) = 0$ and $N \subset B$. For the product space $(\mathbb{R}^r \times \mathbb{R}^s, \mathcal{B}_r \times \mathcal{B}_s, m_r \times m_s)$, if as before we set $\psi_E : \mathbb{R}^s \longrightarrow [0, \infty]$ (for $E \in \mathcal{B}_r \times \mathcal{B}_s$) to be $y \longmapsto m_r(E^y)$, then $\psi_{B \times \mathbb{R}^s} = m_r(B) \chi_{\mathbb{R}^s} = 0$. Thus $m_r \times m_s (B \times \mathbb{R}^s) = \int_{\mathbb{R}^s} \psi_{B \times \mathbb{R}^s} \, dm_s = 0$, i.e., $m_k(B \times \mathbb{R}^s) = 0$. Since $m_k$ is complete and $N \times \mathbb{R}^s \subset B \times \mathbb{R}^s$, we are done. q.e.d.

Example: Let $E = \{x\} \times B$, $x \in \mathbb{R}^r$, $B \subseteq \mathbb{R}^s$, $B \notin \mathcal{L}_s$. Then $E \notin \mathcal{L}_r \times \mathcal{L}_s$ (for otherwise $E_x = B$ would be in $\mathcal{L}_s$). Let $S = \{x\} \times \mathbb{R}^s$. Since $\varphi_S = \mu(\{x\}) \chi_{\mathbb{R}^s} = 0$, therefore $m_k(S) = m_r \times m_s(S) = 0$. Since $E \subset S$ and $m_k$ is complete, this means $E \in \mathcal{L}_k$.

The above example shows that $\mathcal{L}_r \times \mathcal{L}_s \neq \mathcal{L}_k$. However the $\sigma$-algebras $\mathcal{B}_r \times \mathcal{B}_s = \mathcal{B}_k$, $\mathcal{L}_r \times \mathcal{L}_r$, and $\mathcal{L}_k$ are intimately related.

Theorem: $(\mathbb{R}^k, \mathcal{L}_k, m_k)$ is the completion of $(\mathbb{R}^k, \mathcal{L}_r \times \mathcal{L}_s, m_r \times m_s)$.

Proof:

We will first show that

$$\mathcal{B}_k \subset \mathcal{L}_r \times \mathcal{L}_s \subset \mathcal{L}_k.$$

Since $\mathcal{B}_k = \mathcal{B}_r \times \mathcal{B}_s$, and $\mathcal{B}_r \subset \mathcal{L}_r$, $\mathcal{B}_s \subset \mathcal{L}_s$, it is clear that $\mathcal{B}_k \subset \mathcal{L}_r \times \mathcal{L}_s$.

Let $E \in \mathcal{L}_r$. Then $\exists$ an $F_\sigma$-set $A$, a $G_\delta$-set $B$ such that

$$A \subset E \subset B$$

and $m_r(B \sim A) = 0$.

It follows that $E \sim A$ is an $m_r$-null set, as is $B \sim E$. From Lemma 3 it follows that $(E \sim A) \times \mathbb{R}^s \in \mathcal{L}_k$. Since $A \in \mathcal{B}_r$ and $A \times \mathbb{R}^s \in \mathcal{B}_r \times \mathcal{B}_s = \mathcal{B}_k \subset \mathcal{L}_k$, we have

$$E \times \mathbb{R}^s = A \times \mathbb{R}^s \cup (E \sim A) \times \mathbb{R}^s \in \mathcal{L}_k.$$

Thus $E \times \mathbb{R}^s \in \mathcal{L}_k$ $\forall E \in \mathcal{L}_r$. Similarly (or by symmetry) $\mathbb{R}^r \times D \in \mathcal{L}_k$ $\forall D \in \mathcal{L}_s$. This yields

$$E \times D = (E \times \mathbb{R}^s) \cap (\mathbb{R}^r \times D) \in \mathcal{L}_k \qquad \forall\, E \in \mathcal{L}_r,\ D \in \mathcal{L}_s.$$

It follows that

$$\mathcal{L}_r \times \mathcal{L}_s \subset \mathcal{L}_k.$$

Let $Z \in \mathcal{L}_r \times \mathcal{L}_s$. Then $Z \in \mathcal{L}_k$. Hence we can find $A, B$ in $\mathcal{B}_k$ such that $m_k(A - B) = 0$ and $A \subset Z \subset B$. Thus $m_k(A) \le m_k(Z) \le m_k(B)$ & $m_r \times m_s(A) \le m_r \times m_s(Z) \le m_r \times m_s(B)$. Now $A, B \in \mathcal{B}_k$ hence $m_r \times m_s(A) = m_k(A) = m_k(B) = m_r \times m_s(B)$. We thus have

$$\begin{array}{ccccc} m_r \times m_s(A) & \le & m_r \times m_s(Z) & \le & m_r \times m_s(B) \\ \| & & & & \| \\ m_k(A) & \le & m_k(Z) & \le & m_k(B) \end{array}$$

In other words

$$m_r \times m_s(Z) = m_k(Z)$$

for every $Z \in \mathcal{L}_r \times \mathcal{L}_s$.

Since $m_k$ is the completion of $(\mathcal{B}_k, m_r \times m_s)$ and $\mathcal{B}_k \subset \mathcal{L}_r \times \mathcal{L}_s \subset \mathcal{L}_k$, it is clear from the above that $(\mathcal{L}_k, m_k)$ is the completion of $(\mathcal{L}_r \times \mathcal{L}_s, m_r \times m_s)$. q.e.d.

## Two Lemmas on Completions

**Lemma A:** Let $(X, \mathcal{M}, \mu)$ be a measure space and $(X, \hat{\mathcal{M}}, \hat{\mu})$ be its completion. If $f$ is $\hat{\mathcal{M}}$-m'ble then $\exists$ an $\mathcal{M}$-m'ble function $g$ such that $f = g$ a.e. $[\hat{\mu}]$.

**Proof:** We can reduce, in the usual way, to the situation where $f \geq 0$. In this case we have simple m'ble functions $s_n, n \in \mathbb{N}$ s.t.

$$0 = s_1 \leq s_2 \leq \ldots \leq s_n \uparrow f.$$

Now

$$f = \sum_{n=1}^{\infty} (s_{n+1} - s_n)$$

and since each $s_{n+1} - s_n$ can be written as a finite sum $\sum_k c_{nk} \chi_{D_{nk}}$, $c_{nk} \geq 0$, $D_{nk} \in \hat{\mathfrak{M}}$, therefore we can write

$$f = \sum_n a_n \chi_{E_n} \qquad a_n \geq 0,\ E_n \in \hat{\mathfrak{M}}.$$

We can find $A_n, B_n \in \mathfrak{M}$ s.t. $A_n \subset E_n \subset B_n$, $\mu(B_n - A_n) = 0$.

Define

$$g = \sum_n a_n \chi_{A_n}.$$

Then the set $N = \{x \in X \mid f(x) \neq g(x)\} \subseteq \bigcup_n (B_n - A_n)$ and therefore $\hat{\mu}(N) = 0$. Thus $f = g$ a.e. $[\hat{\mu}]$.

**Lemma B:** Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete, $\sigma$-finite measure spaces. Let $(X \times Y, \mathfrak{M}, \lambda)$ be the completion of $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times \nu)$. If $h$ is an $\mathfrak{M}$-m'ble function on $X \times Y$ and $h = 0$ a.e. $[\lambda]$, then $\exists\, N \in \mathcal{S}$, $\mu(N) = 0$ such that for each $x \in X - N$ there exists $F(x) \in \mathcal{T}$ with the property that $\mu(F(x))$ and $h_x(y) = 0$ for all $y \in Y - F(x)$.

Remark: Note that this means $h_x$ is $\mathcal{T}$-m'ble for almost every $x \in X$, for $\nu$ is a complete measure on $\mathcal{T}$, and hence a function which is m'ble outside a $\nu$-null set is measurable on all of $Y$.

**Proof:** There exists $P \in \mathcal{M}$, $\lambda(P) = 0$ such that $h(x,y) = 0$ for every $(x,y) \in X \times Y - P$. Since $(\mathcal{M}, \lambda)$ is the completion of $(\mathcal{S} \times \mathcal{T}, \mu \times \nu)$, there exists $Q \in \mathcal{S} \times \mathcal{T}$, $\mu \times \nu(Q) = 0$, such that $P \subset Q$. For $x \in X$, let, as before, $\varphi_Q(x) = \nu(Q_x)$. Then

$$\int_X \varphi_Q \, d\mu = (\mu \times \nu)(Q) = 0.$$

Since $\varphi_Q \geq 0$, this means $\varphi_Q = 0$ a.e. $[\mu]$. In other words $\exists$ a $\mu$-null set $N$ s.t. for $x \in X - N$, $\varphi_Q(x) = 0$, i.e., $\nu(Q_x) = 0$. Now $P_x \subset Q_x$ and since $\nu(Q_x) = 0$ and $\nu$ is complete, $P_x \in \mathcal{T}$ and $\nu(P_x) = 0$. For $y \in Y - P_x$ we have $h_x(y) = 0$, by our choice of $P$. Set $S(x) = P_x$. Then $N$ and $\{S(x)\}_{x \in X}$ satisfy the conclusions of our Theorem.

q.e.d.