October 30, 2018 Lecture 21

Fubini's Theorem

As in the last lecture $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Recall the following:

Notation: If $f \geq 0$ is $\mathcal{S}\times\mathcal{T}$-measurable then last time we defined

$$\varphi_f : X \longrightarrow [0,\infty] \quad \text{and} \quad \psi_f : Y \longrightarrow [0,\infty]$$

by

$$\varphi_f(x) = \int_Y f_x \, d\nu \qquad (x \in X)$$

and

$$\psi_f(y) = \int_X f^y \, d\mu \qquad (y \in Y).$$

Remark: Recall in the last lecture we had defined φ_E and ψ_E for $E \in \mathcal{S}\times\mathcal{T}$ and shown φ_E is $\mathcal{T}$-m'ble and ψ_E is $\mathcal{S}$-m'ble. It is straightforward to see that

$$\varphi_E = \varphi_{\chi_E} \quad \text{and} \quad \psi_E = \psi_{\chi_E} \qquad (E \in \mathcal{S}\times\mathcal{T})$$

where on the L.S. of each of the two eqns above we are using the old notation from the last lecture and on the right side the notation just given. We will continue to use the symbols φ_E and ψ_E for φ_{χ_E} and ψ_{χ_E}. As we observed last time (partly by defn of $\mu\times\nu$)

$$\int_X \varphi_E \, d\mu = \mu\times\nu(E) = \int_Y \psi_E \, d\nu \qquad \text{——— (1)}$$

Tonelli's Theorem: Let $f: X\times Y \longrightarrow [0,\infty]$ be an $\mathcal{S}\times\mathcal{T}$-m'ble function. Then

(a) φ_f is $\mathcal{S}$-m'ble and ψ_f is $\mathcal{T}$-mble.

(b) $\int_X \varphi_f \, d\mu = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \psi_f \, d\nu$,

or equivalently,

$$\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(x)$$

Proof: We have already proved a special case of Tonelli's theorem, namely the case of $f = \chi_E$ for $E \in \mathcal{S}\times\mathcal{T}$ (see (1) and the comments above it). By taking finite non-negative linear combinations of these characteristic functions we get:

$$\left.\begin{array}{l} \varphi_s \text{ is } \mathcal{S}\text{-m'ble}, \ \psi_s \text{ is } \mathcal{T}\text{-m'ble and} \\ \displaystyle\int_X \varphi_s \, d\mu = \int_{X\times Y} s \, d(\mu\times\nu) = \int_Y \psi_s \, d\nu \\ \text{for all simple non-negative real-valued } \mathcal{S}\times\mathcal{T}\text{-m'ble functions } s \end{array}\right\} \quad \text{——} \ (2)$$

In other words, Tonelli is true for simple $\mathcal{S}\times\mathcal{T}$ non-negative functions.

If f is as in the statement of the Thm, we can find

$$0 \le s_1 \le s_2 \le \dots, \quad s_n \uparrow f$$

with $s_n : X\times Y \longrightarrow [0,\infty)$ simple $\mathcal{S}\times\mathcal{T}$-m'ble. By MCT

$$\varphi_{s_n} \uparrow \varphi_f \qquad \psi_{s_n} \uparrow \psi_f$$

whence φ_f is $\mathcal{S}$-m'ble and ψ_f is $\mathcal{T}$-m'ble (see (2) above).

By (2) again we have

$$\int_X \varphi_{s_n}\, d\mu = \int_{X\times Y} s_n \, d(\mu\times\nu) = \int_Y \psi_{s_n}\, d\nu \qquad (n\in\mathbb{N}).$$

Applying MCT once again, to each of the three sequences of integrals above, we get

$$\int_X \varphi_f\, d\mu = \int_{X\times Y} f\, d\mu\times\nu = \int_Y f\, d\nu$$

as required. q.e.d.

Theorem: Let f be complex m'ble on $(X\times Y, \mathcal{S}\times\mathcal{T})$ and suppose $\varphi_{|f|} \in L^1(\mu)$. Then $f\in L^1(\mu\times\nu)$.

Proof:

This follows immediately from Tonelli for $|f|$. q.e.d.

We now state and prove Fubini's theorem. The statement is in the next page but briefly it says that if f is $\mu\times\nu$-integrable on $X\times Y$ then the iterated integrals exist and the "order of the integration can be reversed" — something very useful in several variables Calculus.

In what follows if f is $(\mathcal{S}\times\mathcal{T})$-m'ble, we use the symbols φ_f & ψ_f as before when they make sense. More precisely

(a) $\varphi_f(x) := \int_Y f_x\, d\nu$ whenever $f_x \in L^1(\nu)$.

(b) $\psi_f(x) := \int_X f^y\, d\mu$ whenever $f^y \in L^1(\mu)$.

Thus if $f^x \in L^1(\nu)$ a.e. $[\mu]$, φ_f is defined on X a.e. $[\mu]$ as a complex function, and if $f^y \in L^1(\mu)$ a.e. $[\nu]$, ψ_f is defined on Y a.e. $[\nu]$ as a cplx fcn.

Theorem (Fubini's Theorem): Suppose f is an $\mathcal{S}\times\mathcal{T}$-m'ble complex function such that $f \in L^1(\mu\times\nu)$. Then

(a) $f_x \in L^1(\nu)$ a.e. $[\mu]$ and $f^y \in L^1(\mu)$ a.e. $[\nu]$.

(b) $\varphi_f \in L^1(\mu)$, $\psi_f \in L^1(\nu)$ (Note: φ_f and ψ_f are m'ble since $\varphi_{|f|}$ & $\psi_{|f|}$ are)

(c) $$\int_X \varphi_f \, d\mu = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \psi_f \, d\nu,$$

i.e.,

$$\boxed{\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y).}$$

Proof:

By Tonelli $\varphi_{|f|} \in L^1(\mu)$, whence $\varphi_{|f|}(x) < \infty$ a.e. $[\mu]$, i.e. $f_x \in L^1(\nu)$ a.e. $[\mu]$. By symmetry $f^y \in L^1[\mu]$ a.e. $[\nu]$. The remaining assertions follow from Tonelli for $(\mathrm{Re} f)^+$, $(\mathrm{Re} f)^-$, $(\mathrm{Im} f)^+$, and $(\mathrm{Im} f)^-$. q.e.d.

Necessity of the hypotheses in Fubini (Examples)

1. Let $X = Y = [0,1]$ with the usual Lebesgue σ-algebra and measure $(\mu = \nu = m)$. Suppose

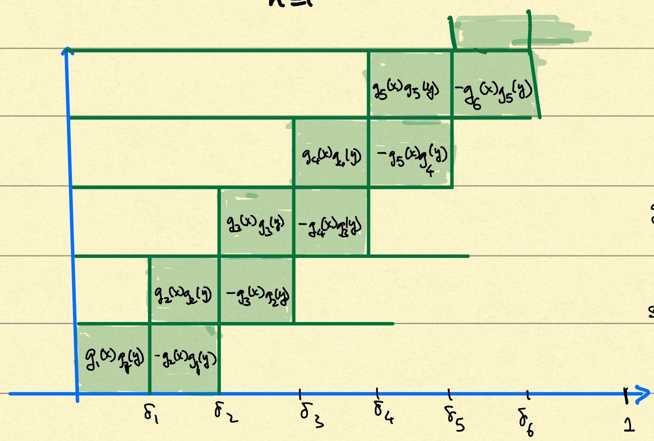
$$0 = \delta_1 < \delta_2 < \cdots < \delta_n \longrightarrow 1.$$

Let g_n be real continuous with support in (δ_n, δ_{n+1}) s.t.

$$\int_0^1 g_n(t)\, dt = 1 \qquad n \in \mathbb{N}.$$

You may assume $g_n \geq 0$ if you wish, though it is not necessary. Define

$$f(x,y) = \sum_{n=1}^{\infty} \{ g_n(x) - g_{n+1}(x) \} g_n(y).$$



The value of f on each grid is given. On the grid squares along the diagonal it is $g_n(x)\, g_n(y)$. On the grid squares in the subdiagonal it is $-g_{n+1}(x)\, g_n(y)$. On the other squares it is zero.

It is clear that

$$\int_0^1 f(x,y)\, dx = g_n(y) - g_n(y) \qquad (\delta_n \le y \le \delta_{n+1})$$
$$= 0.$$

Thus $\quad \int_0^1 f(x,y)\,dx = 0 \quad \forall\ y \in [0,1]. \quad$ —— (1)

Similarly if $x \geq \delta_2$, we have

$$\int_0^1 f(x,y)\, dy = g_2(x)\,(1-1) = 0.$$

However for $0 = \delta_1 \le x \le \delta_2$ we have

$$\int_0^1 f(x,y)\, dy = g_1(x).$$

Thus

$$\int_0^1 f(x,y)\, dy = g_1(x), \quad x \in [0,1] \quad \text{—— (2).}$$

From (1) and (2) we get

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = 0 \neq 1 = \int_0^1 g_1(x)\, dx = \int_0^1 \int_0^1 f(x,y)\, dy\, dx.$$

Note f is continuous on $X \times Y$ except at $(1,1)$. So what went wrong? It is easy to see that

$$\int_0^1 \int_0^1 |f(x,y)|\, dx\, dy = \infty.$$

This is the reason Fubini failed here.

2. Now let $X = Y = [0,1]$, $\mathcal{S}$ = Lebesgue σ-alg, μ = Lebesgue measure; $\mathcal{T} = \mathcal{P}(Y)$, ν = counting measure on $(Y, \mathcal{P}(Y))$.

Note that ν is <u>not</u> σ-finite.

Let $\Delta = \{(x,x) \in X \times Y\}$ ← the diagonal.

We claim Δ is $\mathcal{S} \times \mathcal{T}$-m'ble. Let $n \in \mathbb{N}$. Set

$$I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]. \qquad j = 1, \ldots, n.$$

Let

$$Q_n = I_1 \times I_1 \cup I_2 \times I_2 \cup \ldots \cup I_n \times I_n.$$

Then Q_n is $\mathcal{S} \times \mathcal{T}$-m'ble. Since

$$\Delta = \bigcap_n Q_n$$

Δ is $\mathcal{S} \times \mathcal{T}$-m'ble. It follows that χ_Δ is m'ble.

Now $\int_X f(x,y)\, d\mu(x) = 0$ & $\int_Y f(x,y)\, d\nu(y) = 1$.

Hence

$$\int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y) = 0 \neq 1 = \int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x).$$

This is where the non σ-finiteness of ν plays a role in violating the Fubini principle.

3. (Sieperinski). Assume the continuum hypothesis. In this case there is a well-ordered W and a one-to-one onto W

$$j : [0,1] \longrightarrow W$$

such that $j(x)$ has only a countable number of predecessors for each $x \in [0,1]$. Assume this fact.

Let $X = Y = [0,1]$ with Lebesgue σ-alg & measure. Let

$$Q = \{ (x,y) \in [0,1] \times [0,1] \mid j(x) \leq j(y) \text{ in } W \}$$

So for $x \in X$,

$$Q_x = \{ y \in [0,1] \mid j(x) \leq j(y) \}$$

This means Q_x contains all but a finite or contable number of elements of $[0,1]$. So $\nu(Q_x) = m(Q_x) = 1 \;\forall x \in X$.

On the other hand

$$Q^y = \{ x \in [0,1] \mid j(x) \leq j(y) \}$$

and this is countable at most. Hence $\mu(Q^y) = m(Q^y) = 0$ for every $y \in Y$. Thus

$$\varphi_Q \equiv 1 \quad \text{and} \quad \psi_Q \equiv 0.$$

It follows that $\int_X \varphi_Q \, d\mu = 1 \neq 0 = \int_Y \psi_Q \, d\nu$,

i.e.

$$\int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x) \neq \int_Y \int_X f(x,y) \, d\mu(x) \, d\nu(y)$$

What went wrong?

Answer: f is not $\mathcal{S} \times \mathcal{T}$-measurable.

Fubini for complete measures

We use the notation

$$\mathcal{B}_n = \mathcal{B}(\mathbb{R}^n) \qquad (n \in \mathbb{N}).$$

As before

$\mathcal{L}_n$ = Lebesgue σ-algebra on $\mathbb{R}^n$.

m_n = Lebesgue measure on $(\mathbb{R}^n, \mathcal{L}_n)$.

In what follows k, r, s are positive integers satisfying

$$\boxed{k = r + s.} \quad \text{——— } (*)$$

Product of Borel σ-algebras on $\mathbb{R}^k$:

Lemma 1: $\mathcal{B}_r \times \mathcal{B}_s = \mathcal{B}_k$

Proof:

Let $\mathcal{C} = \{ E \in \mathcal{B}_r \mid E \times \mathbb{R}^s \in \mathcal{B}_k \}$. It is easy to see (check!) that $\mathcal{C}$ is a σ-algebra on $\mathbb{R}^r$ and $\mathcal{C}$ contains all open sets in $\mathbb{R}^r$. It follows that $\mathcal{C} = \mathcal{B}_r$. Thus $E \times \mathbb{R}^s \in \mathcal{B}_k$ for all $E \in \mathcal{B}_r$. By the same reasoning $\mathbb{R}^r \times F \in \mathcal{B}_k$ for every $F \in \mathcal{B}_s$. Since $E \times F = (E \times \mathbb{R}^s) \cap (\mathbb{R}^r \times F)$ therefore all m'ble rectangles in $(\mathbb{R}^k, \mathcal{B}_r \times \mathcal{B}_s)$ lie in $\mathcal{B}_k$ whence

$$\mathcal{B}_r \times \mathcal{B}_s \subset \mathcal{B}_k.$$

On the other hand recall that every open set in $\mathbb{R}^k$ is the countable union of sets of the form $Q = (a_1, b_1) \times \cdots \times (a_n, b_n)$ and such Q clearly lie in $\mathcal{B}_r \times \mathcal{B}_s$. Hence every open set in $\mathbb{R}^k$ lies in $\mathcal{B}_r \times \mathcal{B}_s$, which means

$$\mathcal{B}_k \subset \mathcal{B}_r \times \mathcal{B}_s. \quad \text{q.e.d.}$$