Oct 25, 2018

## Lecture 20

### The Riemann-Lebesgue Lemma for $L^1(\mathbb{T})$ :

As before $u_n(t) = e^{int}$, $t \in \mathbb{Z}$.

Suppose $f \in L^p(\mathbb{T})$, $1 \le p < \infty$. Then clearly $f \cdot u_{-n} \in L^p(\mathbb{T})$ $\forall n \in \mathbb{Z}$. Hence the Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \qquad n \in \mathbb{Z}$$

make sense.

Let $f \in L^p(\mathbb{T})$ and suppose $\varepsilon > 0$ has been given. Then we can find $g \in C(\mathbb{T})$ s.t. $\|f-g\|_p < \varepsilon$ (see Theorem 1.2.1 of Lecture 16 on Oct 9). Further $\exists$ a trigonometric polynomial $P$ s.t. $\|g-P\|_\infty < \varepsilon$. Since $\|g-P\|_p \le \|g-P\|_\infty$, we therefore have

$$\|f-P\| < \varepsilon.$$

Now $P = \sum_{k=-N}^{N} c_k u_k$ and hence for $|n| > N$, $\langle P, u_n \rangle = 0$. Hence for $|n| > N$ we have

$$|\hat{f}(n)| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \{f - P\}\, e^{-int}\, dt \right|$$

$$\le \|f-P\|_p \cdot \|u_{-n}\|_q$$

$$= \|f-P\|_p$$

$$< 2\varepsilon.$$

Thus $\lim_{|n|\to\infty} \hat{f}(n) = 0$. We have thus proven the so-called Riemann-Lebesgue theorem for $L^p(\mathbb{T})$ (see statement on next page). However note that $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, $p \ge 1$, hence it is really an $L^1$ statement.

**Theorem** (The Riemann-Lebesgue Lemma): Let $f \in L^1(\mathbb{T})$. Then

$$\lim_{|n| \to \infty} \hat{f}(n) = 0.$$

Equivalently

$$\lim_{|n| \to \infty} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt = \lim_{|n| \to \infty} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt = 0.$$

We will later prove versions of this for $L^1(\mathbb{R})$ rather than $L^1(\mathbb{T})$. That is a deeper statement.

## Measure Theory for product Spaces

In what follows $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are two $\sigma$-finite measures. Recall that sets of the form $A \times B$, $A \in \mathcal{S}$, $B \in \mathcal{T}$ are called measurable rectangles. The collection $\mathcal{R} = \mathcal{R}(\mathcal{S}, \mathcal{T})$ of finite unions of m'ble rectangles forms an algebra, and in fact every member of $\mathcal{R}$ can actually be written as a finite disjoint union of m'ble rectangles (see HW 7 for all these facts).

**Definition**: $\mathcal{S} \times \mathcal{T} := \sigma(\mathcal{R}(\mathcal{S}, \mathcal{T}))$.

Note: Since $\mathcal{R}$ consists of finite disjoint union of m'ble rectangles, $\mathcal{S} \times \mathcal{T}$ is in fact the $\sigma$-algebra generated by m'ble rectangles.

**Notations:** For $E \subseteq X \times Y$, $x \in X$ and $y \in Y$ define

$$E_x = \{ y \in Y \mid (x,y) \in E \}$$

&

$$E^y = \{ x \in X \mid (x,y) \in E \}.$$

If
$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$
are the two projections then essentially $E_x$ is $p^{-1}(x)$ and $E^y$ is $q^{-1}(y)$.

**Theorem:** Let $E \in \mathcal{S} \times \mathcal{T}$. Then for every $x \in X$ and every $y \in Y$ we have $E_x \in \mathcal{T}$ and $E^y \in \mathcal{S}$.

**Proof:**

Fix $x \in X$. Define

$$\mathcal{M} = \{ E \in \mathcal{S} \times \mathcal{T} \mid E_x \in \mathcal{T} \}.$$

Clearly every m'ble rectangle $A \times B$ lies in $\mathcal{M}$, and hence so does $X \times Y$. Now $(E^c)_x = Y - E_x$ and hence if $E$ is in $\mathcal{M}$ so is $E^c$. If $E = \bigcup_{n=1}^{\infty} E_n$, then $E_x = \bigcup_{n=1}^{\infty} (E_n)_x$ and hence if each $E_n$ is in $\mathcal{M}$ so is $E$. Thus $\mathcal{M}$ is a $\sigma$-subalgebra of $\mathcal{S} \times \mathcal{T}$ containing $\mathcal{R}$. It follows that $\mathcal{M} = \mathcal{S} \times \mathcal{T}$. Thus $E_x \in \mathcal{T}$ $\forall$ $E \in \mathcal{S} \times \mathcal{T}$. Since $x \in X$ is arbitrary, $E_x \in \mathcal{T}$ $\forall$ $x \in X$, $E \in \mathcal{S} \times \mathcal{T}$. By symmetry if $E \in \mathcal{S} \times \mathcal{T}$, and $y \in Y$, then $E^y \in \mathcal{S}$.

**Notation:** Suppose $E \in \mathcal{S} \times \mathcal{T}$. Define $\varphi_E : X \longrightarrow [0,\infty]$ and $\psi_E : Y \longrightarrow [0,\infty]$ by the formulas

$$\varphi_E(x) = \nu(E_x) \qquad x \in X$$

$$\psi_E(y) = \mu(E^y) \qquad y \in Y.$$

This makes sense by the previous Theorem

**Theorem:** For each $E \in \mathcal{S} \times \mathcal{T}$, $\varphi_E$ is $\mathcal{S}$-measurable and $\psi_E$ is $\mathcal{T}$-measurable.

**Proof:** Let $\{X_n\}$ be a $\mathcal{S}$-m'ble partition of $X$ with $\mu(X_n) < \infty$ $\forall n \in \mathbb{N}$, and $\{Y_n\}$ a $\mathcal{T}$-m'ble partition of $Y$ with $\nu(Y_n) < \infty$ $\forall n \in \mathbb{N}$. Let

$$Z_{mn} = X_n \times Y_m, \qquad n, m \in \mathbb{N}.$$

Let

$$\mathcal{C} = \{ E \in \mathcal{S} \times \mathcal{T} \mid \varphi_E \text{ is } \mathcal{S}\text{-measurable} \}.$$

Let $A \times B$ be a m'ble rectangle. Then clearly

$$\varphi_{A \times B} = \nu(B) \cdot \chi_A$$

and hence $\varphi_{A \times B}$ is $\mathcal{S}$-measurable. Moreover, if $R_1, R_2$ are disjoint m'ble rectangles and $Q = R_1 \cup R_2$, then $\varphi_Q = \varphi_{R_1} + \varphi_{R_2}$ whence $Q \in \mathcal{C}$. Thus $\mathcal{R} \subseteq \mathcal{C}$.

If $E_1 \subset E_2 \subset \dots \subset E_n \subset \dots$ is a chain of increasing sets with $E_n \in \mathcal{C}$ $\forall n$, then setting $E = \bigcup E_n$, we see that $(E_n)_x \uparrow E_x$, whence $\nu((E_n)_x) \uparrow \nu(E_x)$, i.e., $\varphi_E = \lim_{n \to \infty} \varphi_{E_n}$, and since each $\varphi_{E_n}$ is $\mathcal{S}$-m'ble so is $\varphi_E$. Thus $E \in \mathcal{C}$.

Suppose we have

$$A \times B \supset E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$$

with $\nu(B) < \infty$ and $E_i \in \mathcal{C}$ $\forall i$. Let $E = \bigcap E_n$. Then we have $E_x = \bigcap_n (E_n)_x$. If $x \notin A$, $(E_n)_x = \emptyset$ $\forall n$, $E_x = \emptyset$, and $\varphi_{E_n}(x) = \varphi_E(x) = 0$. If $x \in A$ then

$$B \supset (E_1)_x \supset (E_2)_x \supset \dots \supset (E_n)_x \supset \dots .$$

Since $\nu(B) < \infty$, $\nu(E_x) = \lim_{n\to\infty} \nu((E_n)_x)$, i.e., in either case ($x \in A$, or $x \notin A$), we have

$$\varphi_E = \lim_{n\to\infty} \varphi_{E_n}.$$

This means $\varphi_E$ is $\mathcal{S}$-m'ble, whence $E \in \mathcal{C}$ in this case.

Let

$$\mathcal{M} = \{ E \in \mathcal{S} \times \mathcal{T} \mid E \cap Z_{mn} \in \mathcal{C} \ \forall\, m, n \in \mathbb{N} \}.$$

Our arguments above show that $\mathcal{M}$ is a monotone class containing $\mathcal{R}$. Since $\mathcal{R}$ is an algebra, by problems in HW7, $\mathcal{M} = \sigma(\mathcal{R}) = \mathcal{S} \times \mathcal{T}$.

Now for $E \in \mathcal{S} \times \mathcal{T}$,

$$\varphi_E = \sum_{m,n} \varphi_{E \cap Z_{mn}}.$$

It follows that $\varphi_E$ is $\mathcal{S}$-m'ble since each $\varphi_{E\cap Z_{mn}}$ is.

By symmetry $\psi_E$ is $\mathcal{T}$-m'ble for every $E \in \mathcal{S} \times \mathcal{T}$.

q.e.d.

**Theorem:** Let $E \in \mathcal{S} \times \mathcal{T}$. Then

$$\int_X \varphi_E \, d\mu = \int_Y \psi_E \, d\nu.$$

Moreover $E \longmapsto \int_X \varphi_E \, d\mu \ (= \int_Y \psi_E \, d\nu)$ is a measure on $\mathcal{S} \times \mathcal{T}$.

**Proof:**

This was Quiz 5. //

**Definition:** With $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ as above ($\mu, \nu$ $\sigma$-finite), the product measure $\mu \times \nu$ on $\mathcal{S} \times \mathcal{T}$ is defined by

$$(\mu \times \nu)(E) = \int_X \varphi_E \, d\mu. \qquad (E \in \mathcal{S} \times \mathcal{T})$$

Note that

$$(\mu \times \nu)(E) = \int_Y \psi_E \, d\nu \qquad\qquad (E \in \mathcal{S} \times \mathcal{T}).$$

Remark: The above defn and the theorem above it really say that

$$\left.\begin{aligned} \int_X \int_Y \chi_E(x,y)\, d\nu(y)\, d\mu(x) &= \int_{X\times Y} \chi_E \, d(\mu\times\nu) \\ &= \int_Y \int_X \chi_E(x,y)\, d\mu(x)\, d\nu(y) \end{aligned}\right\} \forall E \in \mathcal{S}\times\mathcal{T}$$

This is a special case of Tonelli's Theorem which in turn is a special case of Fubini's Theorem.

More Notations: In the above situation if $f$ is a function on $X\times Y$, then for $x \in X$ and $y \in Y$ define $f_x$ on $Y$ and $f^y$ on $X$ by

$$f_x(z) = f(x,z) \qquad\qquad z \in Y$$

and

$$f^y(z) = f(z,y) \qquad\qquad z \in X.$$

Thus for any point $(x,y)$ on $X\times Y$ we have

$$f_x(y) = f^y(x) = f(x,y).$$

Theorem: If $f$ is $\mathcal{S}\times\mathcal{T}$-measurable then for $x \in X$ and $y \in Y$, $f_x$ is $\mathcal{T}$-measurable and $f^y$ is $\mathcal{S}$-measurable.

Proof: If $V$ is a measurable set in the target of $f$, and $E = f^{-1}(V)$, then $E_x = f_x^{-1}(V)$ for $x \in X$ and

$E^y = (f^y)^{-1}(E)$. The conclusion follows from earlier results.

q.e.d.

## Fubini's Theorem

We continue to be in the above situation, i.e., $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are $\sigma$-finite.

**Even more notations:** Let $f$ be $\mathcal{S} \times \mathcal{T}$-m'ble. Suppose $f \geq 0$.

Define

$$\varphi_f : X \longrightarrow [0, \infty] \quad \text{and} \quad \psi_f : Y \longrightarrow [0, \infty]$$

by

$$\varphi_f(x) = \int_Y f_x \, d\nu \qquad (x \in X) \qquad \text{———} \quad (1)$$

and

$$\psi_f(y) = \int_X f^y \, d\mu \qquad (y \in Y). \qquad \text{———} \quad (2)$$

**Tonelli's Theorem:** Let $f : X \times Y \longrightarrow [0, \infty]$ be an $\mathcal{S} \times \mathcal{T}$-m'ble function. Then

(a) $\varphi_f$ is $\mathcal{S}$-m'ble and $\psi_f$ is $\mathcal{T}$-m'ble.

(b) $\int_X \varphi_f \, d\mu = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \psi_f \, d\nu$,

or equivalently,

$$\int_X \int_Y f(x, y) \, d\nu(y) \, d\mu(x) = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \int_X f(x, y) \, d\mu(x) \, d\nu(x)$$

**Note:** From a Remark on the previous page, clearly the statement is true for $f = \chi_E$, where $E \in \mathcal{S} \times \mathcal{T}$. Indeed in that case $\varphi_f = \varphi_E$, $\psi_f = \psi_E$ and these are measurable (w.r.t. $\mathcal{S}$ and $\mathcal{T}$ resp.) and the defn of $\mu \times \nu(E)$ and the previous theorem gives the rest. From here to the general Tonelli theorem follows the usual yoga (simple fcns, approximation by simple fcns, MCT...). We will flesh out details next class.