August 14, 2018

Lecture 2

From the homework problems we see that if $\mathcal{C}$ is any collection of subsets of a non-empty set X, then there exists a σ-algebra $\sigma(\mathcal{C})$ on X such that $\mathcal{C} \subset \sigma(\mathcal{C})$ and if $\mathcal{M}$ is any σ-algebra in X containing $\mathcal{C}$ then $\sigma(\mathcal{C}) \subset \mathcal{M}$. Such a σ-algebra is clearly unique and is called the <u>σ-algebra generated by $\mathcal{C}$</u>.

If X is a topological space, the σ-algebra generated by the open sets of X is called the <u>Borel σ-algebra</u> of X, denoted $\mathcal{B}(X)$, or simply $\mathcal{B}$ (if the context is clear), and members of $\mathcal{B}(X)$ are called <u>Borel sets</u>. A map $f: X \longrightarrow Y$ with Y a topological space is called <u>Borel measurable</u> if it is measurable for $(X, \mathcal{B})$. In other words f is measurable if $f^{-1}(V) \in \mathcal{B}$ for every open set V in Y.

<u>Note</u> : Every continuous function is obviously Borel m'ble.

<u>Theorem</u>: Let Y be a topological space, $(X, \mathcal{M})$ a measurable space, and $f: X \longrightarrow Y$ a map. Let

$$\mathcal{C} = \{ A \subseteq Y \mid f^{-1}(A) \in \mathcal{M} \}.$$

Then

(a) $\mathcal{C}$ is a σ-algebra in Y

(b) If f is measurable then $f^{-1}(A) \in \mathcal{M}$ for every

$A \in \mathcal{B}(Y)$.

(c) If $Y = [-\infty, \infty]$ with its usual topology, then f is measurable if and only if $f^{-1}((a,\infty]) \in \mathcal{M}$ for every a in $\mathbb{R}$.

(The "usual topology on $[-\infty,\infty]$" is given by the following metric: If $a \leq b$, $d(a,b) = \tan^{-1}(b) - \tan^{-1}(a)$, where $\tan^{-1}(-\infty) = -\frac{\pi}{2}$ and $\tan^{-1}(\infty) = \frac{\pi}{2}$.)

(d) If f is measurable, Z a topological space, and $g: Y \longrightarrow Z$ a Borel map, then $g \circ f : X \longrightarrow Z$ is measurable.

Proof:

A special case of (c) is part of your HW and hence we will not prove (c) now.

(a) Note that $f^{-1}(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f^{-1}(A_\alpha)$ and $f^{-1}(Y - A) = X - f^{-1}(A)$, and that $f^{-1}(Y) = X$. From the last relation, it is clear that $Y \in \mathcal{C}$. From the second relation it is clear that $\mathcal{C}$ is closed under complements since $\mathcal{M}$ is. Finally, if $A_1, A_2, \ldots, A_n, \ldots$ are in $\mathcal{C}$ then

$$f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$$

and the right side is in $\mathcal{M}$ since $\mathcal{M}$ is closed under countable unions. This means $\bigcup_n A_n \in \mathcal{C}$, proving (a).

(b) Since f is measurable, by definition all open subsets of Y are members of $\mathcal{C}$. By (a), $\mathcal{C}$ is a σ-algebra. It follows from the fact that $\mathcal{B}(Y)$ is generated by open sets that $\mathcal{B}(Y) \subseteq \mathcal{C}$. Thus $f^{-1}(A) \in \mathcal{M}$ $\forall A \in \mathcal{B}(Y)$.

(c) This essentially one of your HW problems and so the proof is not given here.

(d) This follows from (b). In greater detail, since g is a Borel map, for an open set V in Z, $g^{-1}(V) \in \mathcal{B}(Y)$, and hence by (b), $f^{-1}(g^{-1}(V)) \in \mathcal{M}$, i.e., $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \in \mathcal{M}$.

q.e.d.

Limits of Measurable functions

Recall the definition of $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for a sequence $\{a_n\}$ in $[-\infty,\infty]$ (Note: $-\infty$ and ∞ are allowed as values!). Set

$$\left.\begin{aligned} b_k &= \sup\{a_k, a_{k+1}, \dots\} \\ c_k &= \inf\{a_k, a_{k+1}, \dots\} \end{aligned}\right\} \quad k = 1,2,3,\dots$$

Then $\{b_k\}$ is a decreasing sequencing and $\{c_k\}$ an increasing sequence. Set

$$\beta = \inf\{b_1, b_2, b_3, \dots\} \text{ and } \gamma = \sup\{c_1, c_2, c_3, \dots\}.$$

Note $b_k \longrightarrow \beta$ and $c_k \longrightarrow \gamma$ as $k \to \infty$.

Recall that there is a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$a_{n_j} \longrightarrow \beta \quad \text{as } j \longrightarrow \infty$$

and similarly a subsequence $\{a_{m_r}\}$ of $\{a_n\}$ s.t.

$$a_{m_r} \longrightarrow \gamma \quad \text{as } r \longrightarrow \infty.$$

(Why is this true?)

We write

$$\boxed{\beta = \limsup_{n\to\infty} a_n \quad \text{and} \quad \gamma = \liminf_{n\to\infty} a_n .}$$

β is called the <u>upper limit</u> of $\{a_n\}$ and γ the <u>lower limit</u> of $\{a_n\}$.

Clearly

$$\boxed{\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} (-a_n)}$$

<u>Notations</u>: Suppose $\{f_n\}$ is a sequence of $[-\infty,\infty]$-valued functions on a set X. Define

$$\sup_n f_n : X \longrightarrow [-\infty,\infty]$$

$$\inf_n f_n : X \longrightarrow [-\infty,\infty]$$

$$\limsup_{n\to\infty} f_n : X \longrightarrow [-\infty,\infty]$$

$$\liminf_{n\to\infty} f_n : X \longrightarrow [-\infty,\infty]$$

in the obvious way. For example

$$\left(\sup_n f_n\right)(x) := \sup_n \left(f_n(x)\right) \qquad x \in X$$

and

$$\left(\limsup_{n\to\infty} f_n\right)(x) := \limsup_{n\to\infty} \left(f_n(x)\right) \qquad x \in X.$$

<u>Theorem</u>: If $f_n : X \longrightarrow [-\infty,\infty]$ is m'ble for $n \in \mathbb{N}$, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$ are m'ble.

This means we are assuming that X is a m'ble space.

<u>Proof</u>:

Let $g = \sup_n f_n$. Then

$$g^{-1}((\alpha,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty]).$$

Indeed if $x \in X$ is s.t. $g(x) \in (\alpha,\infty]$ then there is some

f_n s.t. $f_n(x) \in (\alpha, \infty]$ and vice-versa. It follows that $g^{-1}((\alpha,\infty])$ is a m'ble set since each $f_n^{-1}((\alpha,\infty])$ is. By an earlier theorem this means g is m'ble.

Similarly $\inf_n f_n$ is m'ble. Since

$$\limsup_{n\to\infty} f_n = \inf_k \left\{ \sup_{j \geq k} f_j \right\}$$

and

$$\liminf_{n\to\infty} f_n = \sup_k \left\{ \inf_{j \geq k} f_j \right\}$$

it follows that $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are m'ble.

<u>Corollary 1</u> : The limit of every pointwise convergent sequence of complex m'ble functions is m'ble.

(<u>Note</u> : By defn a complex m'ble function cannot take values $-\infty$ or ∞. A complex m'ble function takes values in $\mathbb{C}$ and is m'ble with respect to the topology of $\mathbb{C}$.)

<u>Proof</u> : Suppose $\{f_n\}$ is a sequence of complex m'ble functions and for each n, $f_n = u_n + iv_n$ where u_n and v_n are real valued. Then we have seen that u_n and v_n are m'ble. Now suppose f_n converges pointwise to a complex valued function f. Write $f = u + iv$, $u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$. Then $u_n \to u$ and $v_n \to v$ pointwise as $n \to \infty$. Since $u = \limsup u_n$ ($= \liminf u_n$) u is m'ble. Similarly v is m'ble.

q.e.d.

For any functions f from a set X to $[-\infty,\infty]$, we define

$$\boxed{f^+ = \max\{f,0\} \text{ and } f^- = -\min\{f,0\}.}$$

We then have

$$f^+ \geq 0, \quad f^- \geq 0$$

$$|f| = f^+ + f^-$$

$$f = f^+ - f^-.$$

The functions f^+ and f^- are called the positive and negative parts of f respectively.

There are many ways of writing f as the difference of two non-negative functions, but the representation $f = f^+ - f^-$ is minimal in the following sense.

Proposition: If $f = g - h$, $g \geq 0$, $h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.

Proof: We have $f \leq g$ which means $\max\{f,0\} \leq g$. Similarly $\max\{-f,0\} \leq h$, but $\max\{-f,0\} = -\min\{f,0\}$ and hence $f^- \leq h$. q.e.d.

Corollary 2 (to the Theorem): If $f: X \longrightarrow [-\infty,\infty]$ and $g: X \longrightarrow [-\infty,\infty]$ are m'ble then so are $\max\{f,g\}$ and $\min\{f,g\}$. In particular f^+ and f^- are m'ble.

Proof: $\max\{f,g\} = \sup\{f,g,f,g,f,g,\dots\}$ & $\min\{f,g\} = \inf\{f,g,f,g,\dots\}$.

q.e.d.