for $x \in V \setminus \{0\}$) that

$$m \cdot \|x\|_{2,V} \leq \|x\| \leq M \cdot \|x\|_{2,V}$$

as required. q.e.d.

Riesz's Lemma and the unit ball in a normed space

Example: Let $B = \{x \in \ell^2 \mid \|x\|_2 \leq 1\}$ be the unit ball in ℓ^2 and for $i \in \mathbb{N}$ set

$$e_i = \langle 0, \dots, 0, \underset{\substack{\uparrow \\ i^{\text{th}} \text{ spot}}}{1}, 0, \dots \rangle .$$

In other words e_i is the sequence $\{s_n\}$ where $s_n = \chi_{\{i\}}(n)$. Then $e_i \in \ell^2$ and $\|e_i\|_2 = 1$, whence $e_i \in B$. It is clear that

$$\|e_i - e_j\|_2 = \sqrt{2} \qquad i \neq j, \quad i, j \in \mathbb{N}.$$

It follows that $\{e_n\}$ is a sequence in B which has no convergent subsequence. Thus B is not compact !

The above is a special case of a general phenomenon. If X is a normed linear space (n.l.s. for short), then the unit ball in X is compact if and only if X is finite dimensional.

To prove this we need Riesz's Lemma (see below). It is a way of finding an approximate complement to a closed subspace of a n.l.s.

Theorem (Riesz's Lemma): Let X be a n.l.s and $U \subset X$ a proper closed subspace (i.e. U is closed & $U \neq X$). Then for every δ s.t. $0 < \delta < 1$, there exists $x_\delta \in X$, $\|x_\delta\| = 1$ such that

$$\|x_\delta - u\| > 1 - \delta \qquad \forall\, u \in U.$$

Proof:

Pick $x \in X \setminus U$. Let

$$d = \inf\{\|x - u\| : u \in U\}.$$

Since U is closed, $d > 0$.

Suppose we are given δ s.t. $0 < \delta < 1$. By defn of d there exists $u_\delta \in U$ such that

$$d \leq \|x - u_\delta\| < \frac{d}{1-\delta} \qquad \text{———} \quad (*)$$

Let

$$x_\delta = \frac{x - u_\delta}{\|x - u_\delta\|}.$$

Then $\|x_\delta\| = 1$ and for $u \in U$ we have

$$\|x - u\| = \frac{1}{\|x - u_\delta\|} \, \| x - (u_\delta + \|x - u_\delta\| \cdot u)\|$$

$$\geq \frac{d}{\|x - u_\delta\|} \qquad (\text{since } u_\delta + \|x - u_\delta\| \cdot u \in U)$$

$$> 1 - \delta \qquad (\text{by } (*)).$$

q.e.d.

An important consequence is the following.

Theorem: Let X be a n.l.s and let $B = \{x \in X \mid \|x\| \leq 1\}$. Then B is compact if and only if X is finite dimensional.

Proof:

Since all norms on a finite dimensional vector space are equivalent, if X is finite dim'l B is compact.

Now suppose X is infinite dim'l. Pick $x_1 \in X$, $\|x_1\| = 1$ and let $U_1 = \{\lambda x \mid \lambda \in \mathbb{C}\}$. Since U_1 is finite dim'l and all norms are equivalent on U_1, U_1 is a Banach space with the norm inherited from X, and hence U_1 is closed in X. By Riesz's Lemma, by picking $\delta = \frac{1}{2}$, we can find x_2, $\|x_2\| = 1$ s.t. $\|x_2 - u\| > \frac{1}{2}$ $\forall u \in U_1$. Suppose we have found $x_1, x_2, \dots, x_n$, $n \geq 2$, $\|x_i\| = 1$, $i = 1, \dots, n$ such that $\|x_i - x_j\| > \frac{1}{2}$ for $i \neq j$, $i, j \in \{1, \dots, n\}$. Let U_n be the span of $x_1, x_2, \dots, x_n$. Then U_n is finite dim'l and hence Banach (all norms on U_n are equivalent) and hence closed in X, and $U_n \neq X$ since X is infinite dim'l. By Riesz's Lemma we can find $x_{n+1} \in X$, $\|x_{n+1}\| = 1$ such that $\|x_{n+1} - u\| > \frac{1}{2}$ $\forall u \in U_n$.

Thus we have a sequence $\{x_n\}$, $\|x_n\| = 1$ $\forall n$, s.t.

$$\|x_n - x_m\| > \frac{1}{2} \quad n \neq m, \quad n, m \in \mathbb{N}.$$

It follows that $\{x_n\}$ has no convergent subsequence. Since $\{x_n\}$ is a sequence in B, B cannot be compact.

q.e.d.