October 23, 2018

# Lecture 19

From Lecture 18 we know that $\{u_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$ where $u_n$ is given by $u_n(t) = e^{int}$, $t\in\mathbb{R}$, $n\in\mathbb{Z}$.

For $f\in L^2(\mathbb{T})$ and $n\in\mathbb{Z}$ write

$$\boxed{\hat{f}(n) = \langle f, u_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt}$$

Then

$$\hat{f} : \mathbb{Z} \longrightarrow \mathbb{C}$$

is in $\ell^2(\mathbb{Z})$ and since $\{u_n\}$ is an orthonormal basis for $L^2(\mathbb{T})$, the map $f\longmapsto \hat{f}$ gives an isometry from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$. The function $\hat{f}$ is called the Fourier transform of $f$ and the term is also used for the map $L^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z})$ given by $f\longmapsto\hat{f}$.

Since $\{u_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis we have

$$\left.\begin{aligned} \|f\|_2^2 &= \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2, \quad f\in L^2(\mathbb{T}) \\ &\text{and} \\ \langle f,g\rangle &= \sum_{n\in\mathbb{Z}} \hat{f}(n)\,\overline{\hat{g}(n)}, \quad f,g\in L^2(\mathbb{T}) \end{aligned}\right\} \text{Parseval's identities}$$

**Lemma:** Let $f \in L^2(\mathbb{T})$. Then
$$\lim_{N\to\infty} \| f - \sum_{n=-N}^{N} \hat{f}(n) u_n \|_2 = 0.$$

**Proof:**

Let $g_N = f - \sum_{n=-N}^{N} \hat{f}(n) u_n$, $N \in \mathbb{N}$.

Then
$$\hat{g}_N(n) = \begin{cases} 0 & |n| \leq N \\ \hat{f}(n) & |n| > N. \end{cases}$$

It follows (from Parseval) that
$$\| g_N \|^2 = \sum_{|n|>N} |\hat{f}(n)|^2 \quad \text{———} (*)$$

Since $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \|f\|_2^2 < \infty$, the right side of $(*)$ $\to 0$ as $N \to \infty$. q.e.d.

**Fourier Series:** For $f \in L^2(\mathbb{T})$ the series
$$\sum_{n\in\mathbb{Z}} \hat{f}(n) u_n$$

is called the Fourier series of $f$. According to the Lemma the Fourier series of $f$ converges to $f$ in $L^2(\mathbb{T})$. Accordingly one often writes
$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n) u_n$$

with the understanding that convergence is only in $L^2(\mathbb{T})$.

In 1966, Len Carleson proved that Fourier series of a function $f$ in $L^2(\mathbb{T})$ converges pointwise a.e. to $f$. This is a remarkable theorem for which Carleson got the Abel prize in 2006. Later R. Hunt proved the same result for $f$ in $L^p(\mathbb{T})$ for $1<p<\infty$. In the opposite direction, in 1922, a 19 year old Kolmogorov proved that there exists $f \in L^1(\mathbb{T})$ such that its Fourier series diverges a.e. (and a little later he found an $f \in L^1(\mathbb{T})$ whose Fourier series diverged everywhere). We will not be proving any of these results. The question of whether the Fourier series of a function converges pointwise to that function was posed by Lusin and is called Lusin's problem. (Lusin was Kolmogorov's teacher).

## Some Basic Analysis & Topology

**Lemma:** Let $X$ be a Hausdorff space and $K$ a compact subset of $X$.

(a) $K$ is closed in $X$

(b) If $C$ is a closed subset of $K$, then $C$ is compact.

(c) If $X$ is a metric space $K$ is bounded in $X$.

**Proof:**

(a) Let $x \in X \setminus K$. For each $y \in K$, we can find open sets $U_y$ and $V_y$ with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. Then

$\{V_y \mid y \in K\}$ is an open cover of $K$, and hence there is a finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$ of $K$. Then $U = U_{y_1} \cap \dots \cap U_{y_n}$ is an open nbhd of $x$ and $U \subseteq X \setminus K$. This proves that $X \setminus K$ is open, i.e. $K$ is closed.

(b) Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $C$ and let $U = X \setminus C$. Then $\mathcal{U}' = \mathcal{U} \cup \{U\}$ is an open cover of $K$. Let $\mathcal{V}$ be a finite subcover of $K$. If $U \notin \mathcal{V}$, then $\mathcal{V} \subset \mathcal{U}$, and we have extracted a finite subcover of $C$ from $\mathcal{U}$. If $U \in \mathcal{V}$, then $\mathcal{W} = \mathcal{V} \setminus \{U\}$ is contained in $\mathcal{U}$ and is a finite cover of $C$. Thus $C$ is compact.

(c) Pick $x_0 \in X$ and consider $B_n = \{x \in X \mid d(x, x_0) < n\}$, $n \in \mathbb{N}$. Then $\{B_n\}$ is an open cover of $K$, $B_n \subset B_{n+1}$, $\forall\, n \in \mathbb{N}$. It follows that $\exists\, n \in \mathbb{N}$ s.t. $K \subset B_n$. Thus $K$ is bounded. q.e.d.

Notation: Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Then

$$I^{(1)} := \left[a, \frac{a+b}{2}\right] \quad \text{and} \quad I^{(2)} := \left[\frac{a+b}{2}, b\right].$$

Lemma: Let $I_1, I_2, \dots, I_n$ be closed bounded intervals in $\mathbb{R}^n$ and let $Q = I_1 \times I_2 \times \dots \times I_n$. Then $Q$ is compact in $\mathbb{R}^n$.

Proof:

Suppose not. Then there exists an open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of $Q$ such that $\mathcal{U}$ has no finite subcover of $Q$. Then one of the $2^n$ subsets

$$Q_{k_1,\dots k_n} = I_1^{(k_1)} \times \dots \times I_n^{(k_n)}, \quad k_i \in \{1,2\},\ i=1,\dots,n$$

does not have a finite subcover from $\mathcal{U}$. Call it $Q^{(1)}$.

Then $\operatorname{diam}(Q^{(1)}) = \frac{1}{2}\operatorname{diam}(Q)$. We can repeat the process of subdivision indefinitely to get "subrectangles"

$Q \supset Q^{(1)} \supset Q^{(2)} \supset Q^{(3)} \supset \dots \supset Q^{(m)} \supset \dots$

such that each $Q^{(m)}$ is a product of $n$ closed intervals and

$$\operatorname{diam} Q^{(m)} = \frac{1}{2^m}\operatorname{diam}(Q).$$

The nested interval theorem then shows that

$$\bigcap_m Q^{(m)} = \{x_0\}$$

for some $x_0 \in Q$. (Indeed, pick $x_m \in Q^{(m)}$. Then $\|x_r - x_s\| \leq \frac{1}{2^s}\operatorname{diam}(Q)$ for $s \leq r$, whence $\{x_m\}$ is Cauchy. Set $x_0 = \lim_{m\to\infty} x_m$. Then $x_0 \in \bigcap_m Q^{(m)}$. If $y \in \bigcap_m Q^{(m)}$, then $x_0, y \in Q^{(m)}$ $\forall m$, whence $\|x_0 - y\| \leq \frac{1}{2^m}\operatorname{diam}(Q)$ for all $m \in \mathbb{N}$, i.e. $x_0 = y$.)

Since $x_0 \in Q$, $\exists\, U_\alpha \in \mathcal{U}$ s.t. $x_0 \in U_\alpha$. Since $\operatorname{diam}(Q^{(m)}) \longrightarrow 0$ as $m \to \infty$ and $x_0 \in Q^{(m)}$ $\forall m$, $\exists$ $m$ s.t. $Q^{(m)} \subset U_\alpha$. However $Q^{(m)}$ has no finite subcover from $\mathcal{U}$. This is a contradiction, whence $Q$ is compact.

q.e.d.

**Theorem (The Heine-Borel Theorem):** A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

**Proof:** We have already proven the "only if" part. We now prove the "if" part. So suppose $C$ is closed an bounded in $\mathbb{R}^n$. Since $C$ is bounded there is a closed rectangle $Q$ containing $C$. By the previous lemma, $Q$ is compact. By the earlier lemma, since $C$ is a closed subset of a compact set, $C$ is compact. q.e.d.

## Finite dimensional spaces

Let $N \in \mathbb{N}$. On $\mathbb{C}^N$ consider

$$\| \, \|_\infty : \mathbb{C}^N \longrightarrow [0,\infty)$$

$$\| \, \|_2 : \mathbb{C}^N \longrightarrow [0,\infty)$$

given by

$$\|x\|_\infty = \max_{1\le i\le N} |x_i| \quad , \quad \|x\|_2 = \left\{ \sum_{i=1}^{N} |x_i|^2 \right\}^{1/2}$$

where $x = (x_1, \dots, x_N)$. Clearly

$$|x_i| \le \left\{ \sum_{i=1}^{N} |x_i|^2 \right\}^{1/2}$$

and

$$\left\{ \sum_{i=1}^{N} |x_i|^2 \right\}^{1/2} \le \left\{ \sum_{i=1}^{n} \left\{ \max_{1\le i\le N} |x_i| \right\}^2 \right\}^{1/2}$$

$$\le \left\{ N \cdot \|x\|_\infty^2 \right\}^{1/2} = \sqrt{N}\, \|x\|_\infty .$$

Thus we have

$$\boxed{\|x\|_\infty \le \|x\|_2 \le \sqrt{N}\,\|x\|_\infty \qquad \forall\, x \in \mathbb{C}^N.} \quad \text{—— (i)}$$

This means $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent norms on $\mathbb{C}^N$, i.e., a sequence of points converges to $x_0 \in \mathbb{C}^N$ under $\|\cdot\|_\infty$ if and only if it converges to $x_0$ under $\|\cdot\|_2$. In particular the identity maps $(\mathbb{C}^N, \|\cdot\|_\infty) \longrightarrow (\mathbb{C}^N, \|\cdot\|_2)$ and $(\mathbb{C}^N, \|\cdot\|_2) \longrightarrow (\mathbb{C}^N, \|\cdot\|_\infty)$ are continuous.

Now let $V$ be any finite dim'l vector space over $\mathbb{C}$ with basis

$$b_1, b_2, \ldots, b_N.$$

Every $x \in V$ can be written uniquely as

$$x = a_1(x) b_1 + \cdots + a_N(x) b_N.$$

Define

$$\|x\|_{\infty,V} = \max_{1 \le i \le N} |a_i(x)| \quad \& \quad \|x\|_{2,V} = \left\{ \sum_{i=1}^N |a_i(x)|^2 \right\}^{1/2}.$$

**Theorem:** Let $V$, $b_1, \ldots, b_N$, $a_1, \ldots, a_N$, $\|\cdot\|_{\infty,V}$, $\|\cdot\|_{2,V}$ etc be as as above. Let $\|\cdot\|$ be any norm on $V$. Then $\exists\, m, M \in \mathbb{R}$, $0 < m < M < \infty$ such that

$$m \|x\|_{2,V} \le \|x\| \le M \cdot \|x\|_{2,V} \qquad \forall\, x \in V.$$

In particular, all norms on a finite dimensional vector space over $\mathbb{C}$ are equivalent.

Note: The statement is also true for finite dim'l vector spaces over $\mathbb{R}$, and the proof is the same as the one we are about to give.

**Proof:**

Let $C = \sum_{i=1}^N \|b_i\|$. Then for $x \in V$

$$\begin{aligned} \|x\| = \Big\| \sum_{i=1}^N a_i(x) b_i \Big\| &\le \sum_{i=1}^N |a_i(x)|\, \|b_i\| \le \|x\|_{\infty,V} \sum_{i=1}^N \|b_i\| \\ &\le C \cdot \|x\|_{\infty,V} \\ &\le C \cdot \|x\|_{2,V}. \quad \text{(from (i))} \end{aligned}$$

Thus

$$\|x\| \leq C \cdot \|x\|_{2,V} \qquad x \in V.$$

This means that the identity map

$$(V, \|\cdot\|_{2,V}) \xrightarrow{id} (V, \|\cdot\|)$$

is continuous. On the other hand, on any normed linear space $(X, \|\cdot\|_X)$, the map $\|\cdot\|_X : X \longrightarrow [0,\infty)$ is continuous for $\big| \|x\|_X - \|y\|_X \big| \leq \|x-y\|_X$ by the $\Delta$-inequality.

By considering the commutative diagram

$$\begin{tikzcd}
(V, \|\cdot\|_{2,V}) \arrow[r, "\text{identity}"] \arrow[dr, "\|\cdot\|"'] & (V, \|\cdot\|) \arrow[d, "\|\cdot\|"] \\
& {[0,\infty)}
\end{tikzcd}$$

we see that $\|\cdot\|$ is a continuous function on $(V, \|\cdot\|_{2,V})$.

Let $f : (V, \|\cdot\|_{2,V}) \longrightarrow [0,\infty)$ be given by

$$f(x) = \|x\| \qquad (x \in V).$$

We have just argued that $f$ is continuous on $(V, \|\cdot\|_{2,V})$. Since $S := \{x \in V \mid \|x\|_{2,V} = 1\}$ is compact (being closed & bdd.) and since $0 \notin S$, we have that $f(S)$ is a compact connected subset of $(0,\infty)$. Thus $\exists\, m, M$ s.t

$$f(S) = [m, M] \subset (0,\infty).$$

We therefore have $m \leq f(s) \leq M \quad \forall\, s \in S$, i.e. $m \leq \|s\| \leq M \quad \forall\, s \in S$. It follows (by setting $s = \frac{x}{\|x\|_{2,V}}$