Oct 23, 2018

I realised today (Nov 3) that I had done this calculation in class and forgotten to put in the notes for Lecture 19.

Theorem: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Proof:

Let $f \in L^2(\mathbb{T})$ be the function

$$f(t) = t^2 \qquad t \in [-\pi, \pi].$$

Then

$$\|f\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^4\, dt = \frac{1}{2\pi}\, 2\left(\frac{\pi^5}{5}\right) = \frac{\pi^4}{5} \quad \text{——— } (1).$$

We have $\displaystyle\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-nt}\, dt \qquad n \in \mathbb{Z}.$

So

$$\hat{f}(0) = \frac{1}{2\pi}\, 2\frac{\pi^3}{3} = \frac{\pi^2}{3}. \quad \text{———— } (2)$$

To compute $\hat{f}(n)$ for $n \notin \mathbb{Z}$ let us compute the indefinite integral $\int t^2 e^{xt}\, dt$ for $x \neq 0$.

$$\int t^2 e^{xt}\, dt = \frac{t^2 e^{xt}}{x} - \frac{2}{x}\int t\, e^{xt}\, dt$$

$$= \frac{t^2 e^{xt}}{x} - \frac{2}{x}\left\{\frac{t e^{xt}}{x} - \frac{1}{x}\int e^{xt}\, dt\right\}$$

$$= \frac{t^2 e^{xt}}{x} - \frac{2t e^{xt}}{x^2} + \frac{2}{x^3}\, e^{xt} + C.$$

This means

$$\int t^2 e^{-int}\,dt = -\frac{t^2 e^{-int}}{in} + \frac{2t e^{-int}}{n^2} - \frac{2}{in^3} e^{-int} + C$$

for $n \in \mathbb{Z} \sim \{0\}$

An easy computation (using the fact that $e^{-in\pi} = e^{-in(-\pi)}$) gives:

$$\left.\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\left[\frac{2t\, e^{-int}}{n^2}\right]_{t=-\pi}^{t=\pi} = \frac{1}{2\pi}\, 2\, \frac{2\pi\, e^{-in\pi}}{n^2} \\ &= \frac{2}{n^2} e^{-in\pi}\end{aligned}\right\} n \in \mathbb{Z} \sim \{0\} \quad \text{— (3)}$$

From (2) and (3), using Parseval's identity, we get

$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$$

$$= |\hat{f}(0)|^2 + \sum_{n \in \mathbb{Z} \sim \{0\}} |\hat{f}(n)|^2$$

$$= \frac{\pi^4}{9} + 2 \sum_{n=1}^{\infty} \frac{4}{n^4}$$

i.e. $$\|f\|_2^2 = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \text{——— (4)}$$

Equating the right sides of (4) and (1) we get

$$\frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5}$$

i.e. $$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$, as required.

q.e.d.