October 23, 2018 Lecture 19

From Lecture 18 we know that $\{u_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$ where u_n is given by $u_n(t) = e^{int}$, $t\in\mathbb{R}$, $n\in\mathbb{Z}$.

For $f\in L^2(\mathbb{T})$ and $n\in\mathbb{Z}$ write

$$\hat{f}(n) = \langle f, u_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

Then

$$\hat{f} : \mathbb{Z} \longrightarrow \mathbb{C}$$

is in $\ell^2(\mathbb{Z})$ and since $\{u_n\}$ is an orthonormal basis for $L^2(\mathbb{T})$, the map $f\longmapsto\hat{f}$ gives an isometry from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$. The function $\hat{f}$ is called the Fourier transform of f and the term is also used for the map $L^2(\mathbb{T})\longrightarrow\ell^2(\mathbb{Z})$ given by $f\longmapsto\hat{f}$.

Since $\{u_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis we have

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2, \qquad f\in L^2(\mathbb{T})$$

and

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)\,\overline{\hat{g}(n)}, \qquad f,g\in L^2(\mathbb{T})$$

Parseval's identities

Lemma: Let $f \in L^2(\mathbb{T})$. Then

$$\lim_{N\to\infty} \| f - \sum_{n=-N}^{N} \hat{f}(n) u_n \|_2 = 0.$$

Proof:

Let $g_N = f - \sum_{n=-N}^{N} \hat{f}(n) u_n \quad , \quad N \in \mathbb{N}.$

Then

$$\hat{g}_N(n) = \begin{cases} 0 & |n| \le N \\ \hat{f}(n) & |n| > N. \end{cases}$$

It follows (from Parseval) that

$$\| g_N \|^2 = \sum_{|n|>N} |\hat{f}(n)|^2 \qquad \text{———} (*)$$

Since $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \|f\|_2^2 < \infty$, the right side of $(*)$ $\to 0$ as $N \to \infty$. q.e.d.

Fourier Series: For $f \in L^2(\mathbb{T})$ the series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) u_n$$

is called the Fourier series of f. According to the Lemma the Fourier series of f converges to f in $L^2(\mathbb{T})$. Accordingly one often writes

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n) u_n$$

with the understanding that convergence is only in $L^2(\mathbb{T})$.

In 1966, Len Carleson proved that Fourier series of a function f in $L^2(\mathbb{T})$ converges pointwise a.e. to f. This is a remarkable theorem for which Carleson got the Abel prize in 2006. Later R. Hunt proved the same result for f in $L^p(\mathbb{T})$ for $1<p<\infty$. In the opposite direction, in 1922, a 19 year old Kolmogrov proved that there exists $f \in L^1(\mathbb{T})$ such that its Fourier series diverges a.e. (and a little later he found an $f \in L^1(\mathbb{T})$ whose Fourier series diverged everywhere). We will not be proving any of these results. The question of whether the Fourier series of a function converges pointwise to that function was posed by Lusin and is called Lusin's problem. (Lusin was Kolmogrov's teacher).

Some Basic Analysis & Topology

Lemma: Let X be a Hausdorff space and K a compact subset of X.

(a) K is closed in X

(b) If C is a closed subset of K, then C is compact.

(c) If X is a metric space K is bounded in X.

Proof:

(a) Let $x \in X \setminus K$. For each $y \in K$, we can find open sets U_y and V_y with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. Then

$\{V_y \mid y \in K\}$ is an open cover of K, and hence there is a finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$ of K. Then $U = U_{y_1} \cap \dots \cap U_{y_n}$ is an open nbhd of x and $U \subseteq X \setminus K$. This proves that $X \setminus K$ is open, i.e. K is closed.

(b) Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of C and let $U = X \setminus C$. Then $\mathcal{U}' = \mathcal{U} \cup \{U\}$ is an open cover of K. Let $\mathcal{V}$ be a finite subcover of K. If $U \notin \mathcal{V}$, then $\mathcal{V} \subset \mathcal{U}$, and we have extracted a finite subcover of C from $\mathcal{U}$. If $U \in \mathcal{V}$, then $\mathcal{W} = \mathcal{V} \setminus \{U\}$ is contained in $\mathcal{U}$ and is a finite cover of C. Thus C is compact.

(c) Pick $x_0 \in X$ and consider $B_n = \{x \in X \mid d(x, x_0) < n\}$, $n \in \mathbb{N}$. Then $\{B_n\}$ is an open cover of K, $B_n \subset B_{n+1}$, $\forall\, n \in \mathbb{N}$. It follows that $\exists\, n \in \mathbb{N}$ s.t. $K \subset B_n$. Thus K is bounded. q.e.d.

Notation: Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Then

$$I^{(1)} := \left[a, \frac{a+b}{2}\right] \quad \text{and} \quad I^{(2)} := \left[\frac{a+b}{2}, b\right].$$

Lemma: Let $I_1, I_2, \dots, I_n$ be closed bounded intervals in $\mathbb{R}^n$ and let $Q = I_1 \times I_2 \times \dots \times I_n$. Then Q is compact in $\mathbb{R}^n$.

Proof:

Suppose not. Then there exists an open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of Q such that $\mathcal{U}$ has no finite subcover of Q. Then one of the 2^n subsets

$$Q_{k_1,\dots,k_n} = I_1^{(k_1)} \times \dots \times I_n^{(k_n)}, \quad k_i \in \{1,2\},\ i=1,\dots,n$$

does not have a finite subcover from $\mathcal{U}$. Call it $Q^{(1)}$.

Then $\operatorname{diam}(Q^{(1)}) = \frac{1}{2}\operatorname{diam}(Q)$. We can repeat the process of subdivision indefinitely to get "subrectangles"

$Q \supset Q^{(1)} \supset Q^{(2)} \supset Q^{(3)} \supset \dots \supset Q^{(m)} \supset \dots$

such that each $Q^{(m)}$ is a product of n closed intervals and

$$\operatorname{diam} Q^{(m)} = \frac{1}{2^m}\operatorname{diam}(Q).$$

The nested interval theorem then shows that

$$\bigcap_m Q^{(m)} = \{x_0\}$$

for some $x_0 \in Q$. (Indeed, pick $x_m \in Q^{(m)}$. Then $\|x_r - x_s\| \leq \frac{1}{2^s}\operatorname{diam}(Q)$ for $s \leq r$, whence $\{x_m\}$ is Cauchy. Set $x_0 = \lim_{m\to\infty} x_m$. Then $x_0 \in \bigcap_m Q^{(m)}$. If $y \in \bigcap_m Q^{(m)}$, then $x_0, y \in Q^{(m)}$ $\forall\, m$, whence $\|x_0 - y\| \leq \frac{1}{2^m}\operatorname{diam}(Q)$ for all $m \in \mathbb{N}$, i.e. $x_0 = y$.)

Since $x_0 \in Q$, $\exists\, U_\alpha \in \mathcal{U}$ s.t. $x_0 \in U_\alpha$. Since $\operatorname{diam}(Q^{(m)}) \longrightarrow 0$ as $m \to \infty$ and $x_0 \in Q^{(m)}$ $\forall\, m$, $\exists$ m s.t. $Q^{(m)} \subset U_\alpha$. However $Q^{(m)}$ has no finite subcover from $\mathcal{U}$. This is a contradiction, whence Q is compact.

q.e.d.

Theorem (The Heine-Borel Theorem): A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof: We have already proven the "only if" part. We now prove the "if" part. So suppose C is closed an bounded in $\mathbb{R}^n$. Since C is bounded there is a closed rectangle Q containing C. By the previous lemma, Q is compact. By the earlier lemma, since C is a closed subset of a compact set, C is compact. q.e.d.

Finite dimensional spaces

Let $N \in \mathbb{N}$. On $\mathbb{C}^N$ consider

$$\|\ \|_\infty : \mathbb{C}^N \longrightarrow [0,\infty)$$

$$\|\ \|_2 : \mathbb{C}^N \longrightarrow [0,\infty)$$

given by

$$\|x\|_\infty = \max_{1 \le i \le N} |x_i| \quad , \quad \|x\|_2 = \left\{ \sum_{i=1}^{N} |x_i|^2 \right\}^{1/2}$$

where $x = (x_1, \ldots, x_N)$. Clearly

$$|x_i| \le \left\{ \sum_{i=1}^{N} |x_i|^2 \right\}^{1/2}$$

and

$$\left\{ \sum_{i=1}^{N} |x_i|^2 \right\}^{1/2} \le \left\{ \sum_{i=1}^{n} \left\{ \max_{1 \le i \le N} |x_i| \right\}^2 \right\}^{1/2}$$

$$\le \left\{ N \cdot \|x\|_\infty^2 \right\}^{1/2} = \sqrt{N}\, \|x\|_\infty .$$

Thus we have

$$\boxed{\|x\|_\infty \le \|x\|_2 \le \sqrt{N}\, \|x\|_\infty \qquad \forall\, x \in \mathbb{C}^N .} \quad \text{— (i)}$$

This means $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent norms on $\mathbb{C}^N$, i.e., a sequence of points converges to $x_0 \in \mathbb{C}^N$ under $\|\cdot\|_\infty$ if and only if it converges to x_0 under $\|\cdot\|_2$. In particular the identity maps $(\mathbb{C}^N, \|\cdot\|_\infty) \longrightarrow (\mathbb{C}^N, \|\cdot\|_2)$ and $(\mathbb{C}^N, \|\cdot\|_2) \to (\mathbb{C}^N, \|\cdot\|_\infty)$ are continuous.

Now let V be any finite dim'l vector space over $\mathbb{C}$ with basis

$$b_1, b_2, \ldots, b_N.$$

Every $x \in V$ can be written uniquely as

$$x = a_1(x) b_1 + \cdots + a_N(x) b_N.$$

Define

$$\|x\|_{\infty,V} = \max_{1 \le i \le N} |a_i(x)| \quad \& \quad \|x\|_{2,V} = \left\{ \sum_{i=1}^{N} |a_i(x)|^2 \right\}^{1/2}.$$

Theorem: Let V, $b_1, \ldots, b_N$, $a_1, \ldots, a_N$, $\|\cdot\|_{\infty,V}$, $\|\cdot\|_{2,V}$ etc be as as above. Let $\|\cdot\|$ be any norm on V. Then $\exists\, m, M \in \mathbb{R}$, $0 < m < M < \infty$ such that

$$m \|x\|_{2,V} \le \|x\| \le M \cdot \|x\|_{2,V} \qquad \forall\, x \in V.$$

In particular, all norms on a finite dimensional vector space over $\mathbb{C}$ are equivalent.

Note: The statement is also true for finite dim'l vector spaces over $\mathbb{R}$, and the proof is the same as the one we are about to give.

Proof:

Let $C = \sum_{i=1}^{N} \|b_i\|$. Then for $x \in V$

$$\|x\| = \left\| \sum_{i=1}^{N} a_i(x) b_i \right\| \le \sum_{i=1}^{N} |a_i(x)| \, \|b_i\| \le \|x\|_{\infty,V} \sum_{i=1}^{N} \|b_i\|$$

$$\le C \cdot \|x\|_{\infty,V}$$

$$\le C \cdot \|x\|_{2,V}. \quad \text{(from (i))}$$

Thus

$$\|x\| \le C \cdot \|x\|_{2,V} \qquad x \in V.$$

This means that the identity map

$$(V, \|\cdot\|_{2,V}) \xrightarrow{id} (V, \|\cdot\|)$$

is continuous. On the other hand, on any normed linear space $(X, \|\cdot\|_X)$, the map $\|\cdot\|_X : X \to [0,\infty)$ is continuous for $\big| \|x\|_X - \|y\|_X \big| \le \|x-y\|_X$ by the Δ-inequality.

By considering the commutative diagram

$$\begin{tikzcd}
(V, \|\cdot\|_{2,V}) \arrow[r, "\text{identity}"] \arrow[dr, "\|\cdot\|"'] & (V, \|\cdot\|) \arrow[d, "\|\cdot\|"] \\
 & {[0,\infty)}
\end{tikzcd}$$

we see that $\|\cdot\|$ is a continuous function on $(V, \|\cdot\|_{2,V})$.

Let $f : (V, \|\cdot\|_{2,V}) \longrightarrow [0,\infty)$ be given by

$$f(x) = \|x\| \qquad (x \in V).$$

We have just argued that f is continuous on $(V, \|\cdot\|_{2,V})$. Since $S := \{x \in V \mid \|x\|_{2,V} = 1\}$ is compact (being closed & bdd.) and since $0 \notin S$, we have that $f(S)$ is a compact connected subset of $(0,\infty)$. Thus $\exists\, m, M$ s.t

$$f(S) = [m, M] \subset (0,\infty).$$

We therefore have $m \le f(s) \le M \quad \forall\, s \in S$, i.e. $m \le \|s\| \le M \;\; \forall\, s \in S$. It follows (by setting $s = \frac{x}{\|x\|_{2,V}}$

for $x \in V \setminus \{0\}$) that

$$m \cdot \|x\|_{2,V} \leq \|x\| \leq M \cdot \|x\|_{2,V}$$

as required. q.e.d.

Riesz's Lemma and the unit ball in a normed space

Example: Let $B = \{x \in \ell^2 \mid \|x\|_2 \leq 1\}$ be the unit ball in ℓ^2 and for $i \in \mathbb{N}$ set

$$e_i = \langle 0, \ldots, 0, \underset{\substack{\uparrow \\ i^{\text{th}} \text{ spot}}}{1}, 0, \ldots \rangle .$$

In other words e_i is the sequence $\{s_n\}$ where $s_n = \chi_{\{i\}}(n)$. Then $e_i \in \ell^2$ and $\|e_i\|_2 = 1$, whence $e_i \in B$. It is clear that

$$\|e_i - e_j\|_2 = \sqrt{2} \qquad i \neq j, \quad i, j \in \mathbb{N}.$$

It follows that $\{e_n\}$ is a sequence in B which has no convergent subsequence. Thus B is not compact !

The above is a special case of a general phenomenon. If X is a normed linear space (n.l.s. for short), then the unit ball in X is compact if and only if X is finite dimensional.

To prove this we need Riesz's Lemma (see below). It is a way of finding an approximate complement to a closed subspace of a n.l.s.

Theorem (Riesz's Lemma): Let X be a n.l.s and $U \subset X$ a proper closed subspace (i.e. U is closed & $U \neq X$). Then for every δ s.t. $0<\delta<1$, there exists $x_\delta \in X$, $\|x_\delta\|=1$ such that

$$\|x_\delta - u\| > 1-\delta \qquad \forall\, u \in U.$$

Proof:

Pick $x \in X \setminus U$. Let

$$d = \inf\{\|x-u\| : u \in U\}.$$

Since U is closed, $d>0$.

Suppose we are given δ s.t. $0<\delta<1$. By defn of d there exists $u_\delta \in U$ such that

$$d \le \|x-u_\delta\| < \frac{d}{1-\delta} \qquad (*)$$

Let

$$x_\delta = \frac{x-u_\delta}{\|x-u_\delta\|}.$$

Then $\|x_\delta\|=1$ and for $u \in U$ we have

$$\|x_\delta - u\| = \frac{1}{\|x-u_\delta\|}\,\|x-(u_\delta + \|x-u_\delta\|\cdot u)\|$$

$$\geq \frac{d}{\|x-u_\delta\|} \qquad (\text{since } u_\delta + \|x-u_\delta\|\cdot u \in U)$$

$$> 1-\delta \qquad (\text{by } (*)).$$

q.e.d.

An important consequence is the following.

Theorem: Let X be a n.l.s and let $B = \{x \in X \mid \|x\| \leq 1\}$. Then B is compact if and only if X is finite dimensional.

Proof:

Since all norms on a finite dimensional vector space are equivalent, if X is finite dim'l B is compact.

Now suppose X is infinite dim'l. Pick $x_1 \in X$, $\|x_1\| = 1$ and let $U_1 = \{\lambda x \mid \lambda \in \mathbb{C}\}$. Since U_1 is finite dim'l and all norms are equivalent on U_1, U_1 is a Banach space with the norm inherited from X, and hence U_1 is closed in X. By Riesz's Lemma, by picking $\delta = \frac{1}{2}$, we can find x_2, $\|x_2\| = 1$ s.t. $\|x_2 - u\| > \frac{1}{2}$ $\forall u \in U_1$. Suppose we have found $x_1, x_2, \ldots, x_n$, $n \geq 2$, $\|x_i\| = 1$, $i = 1, \ldots, n$ such that $\|x_i - x_j\| > \frac{1}{2}$ for $i \neq j$, $i, j \in \{1, \ldots, n\}$. Let U_n be the span of $x_1, x_2, \ldots, x_n$. Then U_n is finite dim'l and hence Banach (all norms on U_n are equivalent) and hence closed in X, and $U_n \neq X$ since X is infinite dim'l. By Riesz's Lemma we can find $x_{n+1} \in X$, $\|x_{n+1}\| = 1$ such that $\|x_{n+1} - u\| > \frac{1}{2}$ $\forall u \in U_n$.

Thus we have a sequence $\{x_n\}$, $\|x_n\| = 1$ $\forall n$, s.t.

$$\|x_n - x_m\| > \frac{1}{2} \quad n \neq m, \quad n, m \in \mathbb{N}.$$

It follows that $\{x_n\}$ has no convergent subsequence. Since $\{x_n\}$ is a sequence in B, B cannot be compact.

q.e.d.