LECTURE 18

Date of Lecture: October 16, 2018

All vector spaces are over $\mathbb{C}$.

1. **Complete Orthonormal Sets**

Throughout this section, H is a Hilbert space.

1.1. Last time we stated but did not prove the following theorem.

Theorem 1.1.1. *Let $\{u_\alpha \mid \alpha \in A\}$ be an orthonormal set in H. Then the following conditions are equivalent.*

(a) *$\{u_\alpha \mid \alpha \in A\}$ is a maximal orthonormal set in H.*
(b) *The linear span P of $\{u_\alpha \mid \alpha \in A\}$ is dense in H.*
(c) *For every $x \in H$ we have*

$$\|x\|^2 = \sum_{\alpha\in A} |\widehat{x}(\alpha)|^2.$$

(d) *For every pair of vectors x, y in H we have*

$$\langle x, y\rangle = \sum_{\alpha\in A} \widehat{x}(\alpha)\overline{\widehat{y}(\alpha)}.$$

Proof. Suppose (a) is true. If the linear span P of $\{u_\alpha \mid \alpha \in A\}$ is not dense in H then $Q = \overline{P}^{\perp} \neq 0$, where $\overline{P}$ is the closure of P in H. We can therefore find a vector $u \in Q$ such that $\|u\| = 1$. It is clear that $\{u_\alpha \mid \alpha \in A\} \cup \{u\}$ is also an orthonormal set, contradicting the maximality of $\{u_\alpha \mid \alpha \in A\}$. We have just proved that $(a) \implies (b)$.

Next assume (b) is true. Then (c) is an immediate consequence of Theorem 1.6.1 of Lecture 17.

Now assume (c). We have the *polarisation identity* (see Problem (4) of Quiz 2) for any Hilbert Space $(M, \langle\cdot,\cdot\rangle_M)$ and for any $x, y \in M$.

$$\langle x, y\rangle_M = \frac{1}{4}\left(\|x+y\|_M^2 - \|x-y\|_M^2 + i\|x+iy\|_M^2 - i\|x-iy\|_M^2\right).$$

Apply this to both the Hilbert spaces H and $\ell^2(A)$. Then (d) follows easily from (c).

As for $(d) \implies (a)$, suppose (d) is true and $\{u_\alpha \mid \alpha \in A\}$ is not a maximal orthonormal set. Then there exists a vector $u \in H$, $\|u\| = 1$, such that such that $\langle u, u_\alpha\rangle = 0$ for every $\alpha \in A$. In particular this means $\widehat{u}(\alpha) = 0$ for every $\alpha \in A$. By (d) this means for any $x \in H$,

$$\langle u, x\rangle = \sum_{\alpha\in A} \widehat{u}(\alpha)\overline{\widehat{x}(\alpha)} = 0.$$

Thus $u = 0$ contradicting the fact $\|u\| = 1$. Thus $(d) \implies (a)$. ◻

Definition 1.1.2. An orthonormal set $\{u_\alpha \mid \alpha \in A\}$ in H is said to be a *complete orthonormal set* for H or an *orthonormal basis* of H if it satisfies any of the equivalent conditions of Theorem 1.1.1.

Theorem 1.1.3. *If H is non-zero then it has a complete orthonormal set.*

Proof. Since $H \neq 0$ it has a vector u of norm 1, and $\{u\}$ is an orthonormal set in H. Thus the collection $\mathscr{A}$ of orthonormal sets in H is non-empty. $\mathscr{A}$ has a natural order given by inclusion of sets. If we have a chain $\mathscr{S}$ in $\mathscr{A}$, say $\mathscr{S} = \{S_\lambda \mid \lambda \in \Lambda\}$ where Λ is a totally ordered set, with $S_{\lambda_1} \subset S_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$, then

$$S = \bigcup_{\lambda \in \Lambda} S_\lambda$$

is easily seen to be an orthonormal set. Thus by Zorn's lemma $\mathscr{A}$ has a maximal element $S^* = \{u_\alpha \mid \alpha \in A\}$. By definition S^* is a complete orthonormal set. $\square$

2. Trigonometric Polynomials and $L^2(T)$

Throughout this section we set T equal to the unit circle centred at 0 in $\mathbb{C}$, i.e.

$$T = \{z \in \mathbb{C} \mid |z| = 1\}.$$

By a *periodic* function g on $\mathbb{R}$ we mean a function g which is periodic with period 2π, i.e. g satisfies $g(t + 2\pi) = g(t)$ for every $t \in \mathbb{R}$. We identify functions on T with periodic functions on $\mathbb{R}$ in the usual way. In other words, if $e\colon \mathbb{R} \to T$ is the usual map $t \mapsto e^{it}$, then a function f on T gets identified with $g = f \circ e$. In fact we will often write $f(t)$ for $f(e^{it})$, so that the same symbol is used for the function on T as well as its "lift" to $\mathbb{R}$. It is clear that every periodic function arises from a function on T in a unique way.

2.1. The space $L^2(T)$. A *measurable* function on T will be (for us) a function such that the corresponding periodic function is Lebesgue measurable $\mathbb{R}$. It is not hard to see the following (though we will probably not use it). On $\mathscr{B}(T)$ one has the arc-length measure $d\theta$. Complete $\mathscr{B}(T)$ with respect to this measure to get a σ-algebra $\mathscr{L}(T)$. Call the members of $\mathscr{L}(T)$ Lebesgue measurable sets in T. These are the same as subsets $S \subset T$ such that $e^{-1}(S) \in \mathscr{L}(\mathbb{R})$. Then a measurable function on T is the same as an $\mathscr{L}(T)$-measurable function.

On $(T, \mathscr{L}(T))$ we have the so-called *Haar measure* μ, namely, with m the usual Lebesgue measure on $\mathbb{R}$:

$$\mu(E) = \frac{1}{2\pi} m\Big(e^{-1}(E) \cap [-\pi, \pi]\Big) \qquad (E \in \mathscr{L}(T)). \tag{2.1.1}$$

For $p \in [1, \infty]$, set

$$L^p(T) := L^p(\mu). \tag{2.1.2}$$

In terms of periodic functions, for $1 \leq p < \infty$, $f \in L^p(T)$ if it is measurable and

$$\frac{1}{2\pi} \int_{[-\pi,\pi]} |f|^p \, dm < \infty. \tag{2.1.3}$$

Of course, as usual, members of $L^p(T)$ are really equivalence classes of such f, the equivalence being "equal a.e. $[\mu]$".

We have the standard inner product and norm on $L^2(T)$ and we know it is a Hilbert space. The rest of this lecture is devoted to finding an orthonormal basis for $L^2(T)$.

2.2. Trigonometric polynomials. Consider the periodic functions u_n given by

$$u_n(t) = e^{int} \qquad (t \in \mathbb{R},\, n \in \mathbb{Z}). \tag{2.2.1}$$

Since T is compact and the u_n are continuous on T, we have $u_n \in L^\infty(T)$ whence in $L^2(T)$. If $\|\cdot\|$ and $\langle\, ,\rangle$ represents the norm and inner product in $L^2(T)$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u_n|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt = 1$$

giving

$$\|u_n\| = 1 \qquad (n \in \mathbb{Z}). \tag{2.2.2}$$

Similarly, since $\int_{-\pi}^{\pi} e^{kit}dt = 0$ for $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\langle u_n, u_m\rangle = 0 \qquad (n, m \in \mathbb{Z}, n \neq m). \tag{2.2.3}$$

Thus $\{u_n \mid n \in \mathbb{Z}\}$ is an orthonormal set in $L^2(T)$. In fact it is an orthonormal basis as we shall soon see.

A *trigonometric polynomial* is a finite sum of the form

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt) \qquad (t \in \mathbb{R}). \tag{2.2.4}$$

This can be re-written as

$$f = \sum_{n=-N}^{N} c_n u_n(t). \tag{2.2.5}$$

From (2.2.5) it is clear that trigonometric polynomials are exactly the elements of the linear span of of the orthonormal set $\{u_n \mid n \in \mathbb{Z}\}$ where u_n are as in (2.2.1).

Definition 2.2.6. Let $f, g \in C(T)$. Define a function $f * g$ on $\mathbb{R}$ formula

$$f * g(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)ds \qquad (t \in \mathbb{R}).$$

Note that $f * g$ is periodic (since f is) and hence is a function on T. In fact $f * g \in C(T)$ (a simple exercise, using the uniform continuity of f on the compact space T, and is left to you), but we do not need this fact today. And later we will prove more general statements for a larger class of functions. What we need today is the following pair of simple observations.

Lemma 2.2.7. *Let $f, g \in C(T)$.*

(a) *$f * g = g * f$.*

(b) *With u_n as in* (2.2.1)*, we have*

$$f * u_n = \langle f, u_n\rangle u_n \qquad (n \in \mathbb{Z}).$$

In particular, if g is a trigonometric polynomial then so is $f * g$.

Proof. For part (a), make the change of variables $s^* = t - s$, and use the fact that f and g are periodic to see that integrating the resulting integrand from $-\pi + t$ to $\pi + t$ is the same as integrating from $-\pi$ to π.

The following calculation (for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$) which uses (a), proves (b).

$$f * u_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)u_n(t-s)ds = e^{nt}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-ns}ds = \langle f, u_n\rangle u_n(t).$$

□

2.3. The heuristics of the Dirac delta function. The illustrates a heuristic which helps with the actual construction of a proof that $\{u_n\}$ is a complete orthonormal basis for $L^2(T)$. Those uninterested in heuristics can skip directly to the definition of the approximate identity $\{Q_n\}$ in Subsection **??**

Part of the idea comes from the notion of the *Dirac delta function* introduced by Dirac in quantum mechanics.[1] The periodic version of this is supposed to be a function $\delta(t)$ such that

- For every $-\pi \leq a < 0 < b \leq \pi$,
$$\frac{1}{2\pi}\int_a^b \delta(t)dt = 1.$$
- $\delta(t) = 0$ for $0 < |t| \leq \pi$.

It is clear that there is no such function, for such a function (when restricted to $[-\pi, \pi]$) would be the Radon-Nikodym derivative of the Dirac delta measure δ_0 with respect to the Lebesgue measure m. However, we know that $\delta_0 \perp m$, and so $d\delta_0/dm$ does not exist, i.e. the Dirac delta function does not exist. That said it is a useful notion, and an important guide to our thinking. Suppose, for the sake of discussion, that $\delta(t)$ did indeed exist. If f is a periodic function, continuous at 0, then in a small neighbourhood of 0, f is close to the constant function $f(0)$. In somewhat greater detail, given $\epsilon > 0$, there exists $\gamma > 0$ such that $|f(t) - f(0)| < \epsilon$ for $|t| < \gamma$. Now, by definition of $\delta(t)$, we must have

$$(*) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t) - f(0))\delta(t)dt = \frac{1}{2\pi}\int_{-\gamma}^{\gamma}(f(t) - f(0))\delta(t)dt$$

and from our choice of γ

$$(**) \qquad -\epsilon\frac{1}{2\pi}\int_{-\gamma}^{\gamma}\delta(t)dt \leq \frac{1}{2\pi}\int_{-\gamma}^{\gamma}(f(t) - f(0))\delta(t)dt \leq \epsilon\frac{1}{2\pi}\int_{-\gamma}^{\gamma}\delta(t)dt.$$

Since $\frac{1}{2\pi}\int_{-\gamma}^{\gamma}\delta(t)dt = 1$, $(*)$ and $(**)$ give us $-\epsilon \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t) - f(0))\delta(t)dt \leq \epsilon$, and since $\epsilon > 0$ is arbitrary, we have:

$$(\dagger) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\delta(t)dt = f(0).$$

If g is the function $s \mapsto f(t - s)$, then according to $(\dagger)$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\delta(s)ds = g(0)$, yielding

$$(\ddagger) \qquad f * \delta = f.$$

Suppose further that δ can be approximated by trigonometric polynomial. Then according to Lemma 2.2.7 (b), and $(\ddagger)$, $f \in C(T)$ can be approximated (in some sense) by trigonometric polynomials, giving us a way of showing $\{u_n \mid n \in \mathbb{Z}\}$ is complete as an orthonormal set.

One can make all this rigorous in certain situations in a couple of ways. The function δ is to be interpreted as a *distribution* or a *generalised function* in the sense of Laurent Schwartz.[2] See [VD] for more details.

[1]The history actually goes back to Heaviside, a British Engineer, though Dirac used it in a deeper way.

[2]Sobolev laid some of the foundations in the 1930s before Schwartz reworked it in a systematic way in the late 1940s, an effort which won him the Fields Medal in 1950.

2.4. A specific approximation to δ. Consider the sequence $\{Q_n\}$ of trigonometric polynomials given by

$$Q_n(t) = c_n\left(\frac{1+\cos t}{2}\right)^n \qquad (t \in \mathbb{R},\ n \in \mathbb{N}). \tag{2.4.1}$$

where c_n are so chosen that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Q_n(t)dt = 1 \qquad (n \in \mathbb{N}). \tag{2.4.2}$$

It is clear that

$$Q_n(t) \geq 0 \qquad (t \in \mathbb{R}, n \in \mathbb{N}). \tag{2.4.3}$$

We wish to use the data $\{Q_n\}$ as a proxy for $\delta(t)$ in the sense that for $f \in C(T)$, $f * Q_n \sim f$ for $n \gg 0$. In fact we will show that $\lim_{n\to\infty} \|f * Q_n - f\|_\infty = 0$ (see Lemma 2.4.6). One ideal (but non-acheivable) property of δ that we wish to replicate, perhaps weakly, is the property that $\delta(t) = 0$ if t is non-zero in $[\pi, \pi]$. One obvious formulation is to require that "off a neighbourhood of 0", Q_n converges to 0 uniformly. In greater detail, here is the agenda. For $0 < \gamma \leq \pi$, define

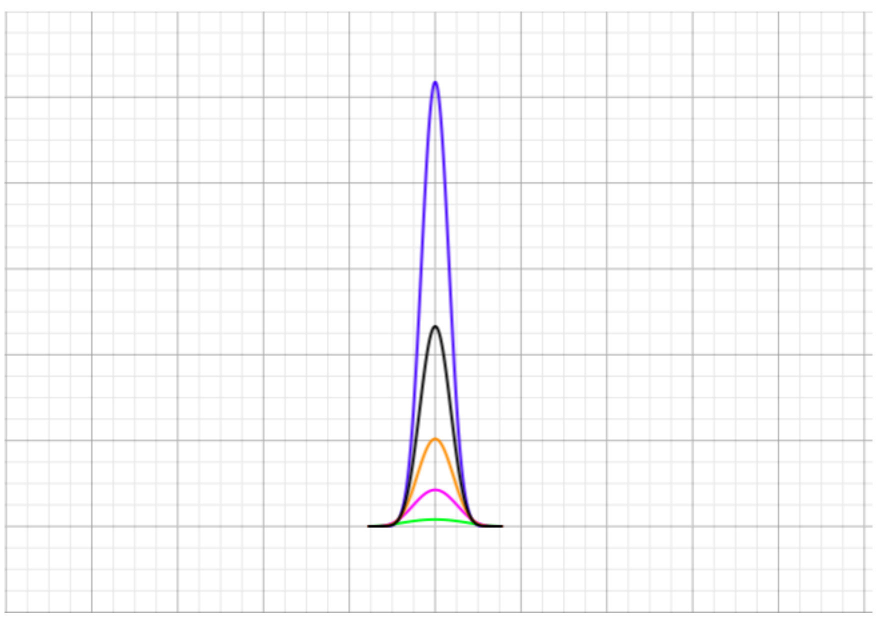
$$\eta_n(\gamma) = \sup_{\delta<|t|\leq\pi} Q(t). \tag{2.4.4}$$

We will show that

$$\lim_{n\to\infty} \eta_n(\gamma) = 0. \tag{2.4.5}$$

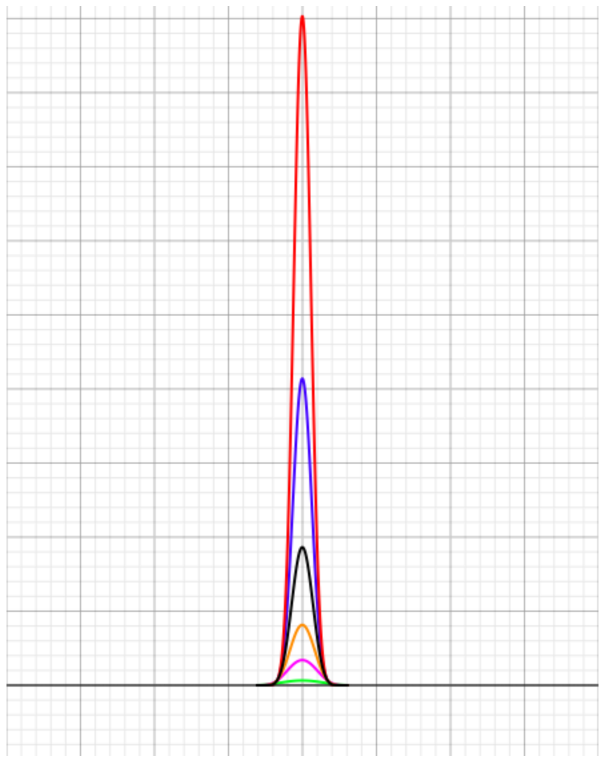
for every $0 < \gamma \leq \pi$.

Here are the graphs of Q_1, Q_2, Q_3, Q_4, and Q_5.



Note that the graphs get narrower, the maximum keeps increasing, while the areas under the curves remain constant at 1. This is what forces $\eta_n(\gamma)$ to approach 0 as $n \to \infty$.

For greater intuition, here are the graphs of Q_1, Q_2, Q_3, Q_4, Q_5, and Q_6.



Any sequence $\{Q_n\}$ satisfying (2.4.2), (2.4.3) and (2.4.5) (where η_n is defined by (2.4.4)) is called an *approximate identity* or a *mollifier* for functions on T.

The proof of (2.4.5) for the sequence in (2.4.1) is easy to prove. By the symmetry of Q_n, (2.4.2) gives, for $n \in \mathbb{N}$,

$$\begin{aligned} 1 &= \frac{c_n}{\pi} \int_0^\pi \left(\frac{1+\cos t}{2} \right)^n dt \\ &\geq \frac{c_n}{\pi} \int_0^\pi \left(\frac{1+\cos t}{2} \right)^n \sin t \, dt \\ &= \frac{2c_n}{\pi(n+1)}. \end{aligned}$$

Thus

$$c_n \leq \frac{\pi(n+1)}{2} \qquad (n \in \mathbb{N}).$$

Let $\gamma \in (0, \pi]$ and fix $n \in \mathbb{N}$. By the symmetry of Q_n, $\eta_n(\gamma) = \sup_{[\eta,\pi]} Q_n(t)$. Since Q_n is decreasing on $[0, \pi]$, $\eta_n(\gamma) = Q_n(\gamma)$. Thus

$$\eta_n(\gamma) = Q_n(\gamma) = c_n \left(\frac{1+\cos\gamma}{2} \right)^n \leq \frac{\pi(n+1)}{2} \left(\frac{1+\cos\gamma}{2} \right)^n.$$

Since $0 < \gamma \leq \pi$, therefore $0 \leq \cos\gamma < 1$, whence $0 < \frac{1+\cos\gamma}{2} < 1$. It follows that $\eta_n(\gamma) \to 0$ as $n \to \infty$ for every $\gamma \in (0, \pi]$. This establishes (2.4.5).

The following Lemma shows that the data $\{Q_n\}$ does approximate the property of the Dirac delta function given in (‡).

Lemma 2.4.6. *Let $f \in C(T)$ and set $P_n = f * Q_n$, $n \in \mathbb{N}$, where $\{Q_n\}$ is as in* (2.4.1). *Then*

$$\lim_{n\to\infty} \|P_n - f\|_\infty = 0.$$

Note: P_n are trigonometric polynomials (cf. Lemma 2.2.7 (b)).

Proof. Pick $\epsilon > 0$. Since T is compact, f is uniformly continuous, and hence there exists $\gamma > 0$ such that $|f(t) - f(s)| < \epsilon$ whenever $|t - s| < \gamma$. For $t \in [-\pi, \pi]$ we therefore have

$$\begin{aligned}\Big|P_n(t) - f(t)\Big| &= \frac{1}{2\pi}\Big|\int_{-\pi}^{\pi} (f(t-s) - f(t))Q_n(s)\,ds\Big| \\ &\le \frac{1}{2\pi}\int_{-\gamma}^{\gamma} |f(t-s) - f(t)|Q_n(s)\,ds \\ &\qquad + \frac{1}{2\pi}\int_{\{\pi\ge|t|\ge\gamma\}} |f(t-s) - f(t)|Q_n(s)\,ds \\ &\le \epsilon + 2\|f\|_\infty \eta_n(\gamma).\end{aligned}$$

This estimate is independent of t by our choice of γ and the definition of $\eta_n(\gamma)$. By (2.4.5) we can find N such that $\eta_n(\gamma) \le \epsilon$ for $n \ge N$, and it follows that $\|P_n - f\|_\infty \le 2\epsilon$ for $n \ge N$. This proves the Lemma. □

2.5. Completeness of the orthonormal set $\{u_n \mid n \in \mathbb{Z}\}$. We are now in a position to show that trigonometric polynomials are dense in $L^2(T)$.

Theorem 2.5.1. *Trigonometric polynomials form a dense subspace of $L^2(T)$. In particular $\{u_n \mid n \in \mathbb{Z}\}$ forms a complete orthonormal set in $L^2(T)$.*

Proof. According to Theorem 1.2.1 of Lecture 16, $C(T)$ is dense in $L^2(T)$, since T is compact. Therefore, given $\epsilon > 0$, it is enough to show that for each $f \in C(T)$ there exists a trigonometric polynomial P such that $\|P - f\| < \epsilon$, where $\|\cdot\|$ is the norm in $L^2(T)$. Now for any $h \in C(T)$ we have,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |h|^2\,dt \le \|h\|_\infty^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} dt = \|h\|_\infty^2.$$

In other words

$$\|h\| \le \|h\|_\infty.$$

Let $f \in C(T)$. By Lemma 2.4.6 (see the "Note" after its statement) given $\epsilon > 0$ we have a trigonometric polynomial P such that $\|P - f\|_\infty < \epsilon$. This means $\|P - f\| \le \|P - f\|_\infty < \epsilon$, which is what we wished to show. □

References

[R] W. Rudin, *Real and Complex Analysis*, (Third Edition), McGraw-Hill, New York, 1987.

[VD] G. van Dijk, *Distribution Theory*, De Gruyter, Berlin/Boston, 2013. (See also `http://www.math.nagoya-u.ac.jp/ richard/teaching/s2017/Gerrit_2013.pdf`.)