LECTURE 17

Date of Lecture: October 11, 2018

All vector spaces are over $\mathbb{C}$.

1. Basics

Here are some proofs of basic facts we have been using.

1.1. The norm of a linear operator.

Let

$$T\colon X \longrightarrow Y$$

be a linear transformation between normed linear spaces. Let

$$A = \left\{ \frac{\|Tx\|}{\|x\|} \,\middle|\, 0 \neq x \in X \right\} = \{\|Tx\| \mid x \in X, \|x\| = 1\},$$

and,

$$B = \{M \in \mathbb{R} \mid \|Tx\| \leq Mx, \forall x \in X\}.$$

If $N \in A$ and M in B it is clear that $N \leq M$. It follows that $\sup_{N\in A} N \leq \inf_{M\in B} M$. On the other hand, clearly $\sup_{N\in A} N \in B$. So

$$\sup_{N\in A} N = \inf_{M\in B} M.$$

The common number above is called the *norm of* T, and written $\|T\|$. A little thought shows that

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|.$$

Definition 1.1.1. T is said to be *bounded* if $\|T\| < \infty$.

Theorem 1.1.2. *T is continuous if and only if it is bounded.*

Proof. Suppose T is continuous. Then it is continuous at 0, and therefore we have $\delta > 0$ such that $|Tx\| < 11$ whenever $\|x\| < \delta$. Then $\|T(\delta x/\|x\|)\| < 1$ for all $x \neq 0$. This means $\|T| \leq 1/\delta < \infty$, i.e. T is bounded.

Suppose now that T is bounded and $x_\circ \in X$. Let $\{x_n\}$ be a sequence in X converging to $x_\circ$. Then

$$\|Tx_n - Tx_\circ\| = \|T(x_n - x_\circ)\| \leq \|T\|\|x_n - x_\circ\| \longrightarrow 0$$

as $n \to \infty$. This proves that T is continuous at $x_\circ$. ☐

1.2. **The Cauchy-Schwarz inequality.**

Theorem 1.2.1. *(Cauchy-Schwarz) Let X be an inner product space. Then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

for every $x, y \in X$, with equality occuring if and only if one of x or y is a scalar multiple of the other.

Proof. Suppose x and y are such that neither is a scalar multiple of the other. Let α be the complex number such that $\alpha\langle x, y\rangle = |\langle x, y\rangle|$, $|\alpha| = 1$. Our hypotheses on x and y ensure that x, y, and $\alpha x - y$ are all non-zero. Therefore

$$\langle \alpha x - y, \alpha x - y\rangle > 0.$$

Expanding this and re-arranging terms, using the fact that $|\alpha| = 1$ and $\alpha\langle x, y\rangle = \bar{\alpha}\langle y, x\rangle = |\langle x, y\rangle|$, we get

$$\|x\|^2 + \|y\|^2 > 2|\langle x, y\rangle|.$$

If t is a positive real number then $u = tx$ and $v = (1/t)y$ are not scalar multiples of each other and $\langle u, v\rangle = \langle x, y\rangle$. The above inequality applied to u and v then gives

$$(*) \qquad t^2\|x\|^2 + \frac{1}{t^2}\|y\|^2 > 2|\langle x, y\rangle| \qquad (t > 0).$$

Let $f(t) =^2 \|x\|^2 + t^{-2}\|y\|^2$, for $t > 0$. Basic Calculus shows that f has a unique minimum at $t_\circ = \sqrt{\|y\|/\|x\|}$. Setting $t = t_\circ$ in $(*)$ we get $2\|x\|\|y\| > 2|\langle x, y\rangle|$, giving the required inequality. $\square$.

1.3. **Orthonormal sets.** For the rest of the lecture, H is a *Hilbert spacee*. A set of vectors $\{u_\alpha \mid \alpha \in A\}$ in H is said to be an *orthonormal set* if $\langle u_\alpha, u_\beta\rangle = 0$ for $\alpha \neq \beta$, $\alpha, \beta \in A$ and $\|u_\alpha\| = 1$, $\alpha \in A$. Note that the map $\alpha \mapsto u_\alpha$ is necessarily a bijective map from A to $\{u_\alpha \mid \alpha \in A\}$.

So suppose $S = \{u_\alpha \mid \alpha \in A\}$ is an orthonormal set. For each $\alpha \in A$ we have

$$\widehat{x}_{S,A}(\alpha) = \langle x, u_\alpha\rangle.$$

Then we get a map

$$\widehat{x}_{S,A} \colon A \longrightarrow \mathbb{C}.$$

From now on we will write $\widehat{x}$ for $\widehat{x}_{S,A}$. It is sometimes called the Fourier transform of x, and $\widehat{x}(\alpha)$ the Fourier coefficient at α associated to x.

Theorem 1.3.1. *Let $\{u_\alpha \mid \alpha \in A\}$ be an orthonormal set and F a finite subset of A. Let M_F be the linear span of u_α for $\alpha \in F$.*

(a) *Let $\varphi \colon A \to \mathbb{C}$ be a function such that* $\operatorname{Supp}\varphi \subset F$. *Then there exists a unique $y \in M_F$, namely*

$$(1.3.1.1) \qquad y = \sum_{\alpha \in F} \varphi(\alpha) u_\alpha,$$

such that $\widehat{y} = \varphi$. Moreover,

$$(1.3.1.2) \qquad \|y\|^2 = \sum_{\alpha \in F} |\varphi(\alpha)|^2.$$

(b) *If* $x \in H$ *and*

$$s_F = \sum_{\alpha \in F} \widehat{x}(\alpha) u_\alpha, \tag{1.3.1.3}$$

then

$$\|x - s_F\| < \|x - s\| \tag{1.3.1.4}$$

for every $s \in M_F$ *except when* $s = s_F$ *and*

$$\sum_{\alpha \in F} |\widehat{x}(\alpha)|^2 \leq \|x\|^2. \tag{1.3.1.5}$$

Proof. Part (a) is obvious. The only slightly non-trivial part is the use of the identity $\|a+b\|^2 = \|a|^2 + \|b\|^2$ for $a \perp b$, $a, b \in H$. Uniqueness is a direct consequence of (1.3.1.2).

For part (b), note that if $\alpha \in F$ then $\langle x - s_F, u_\alpha \rangle = \widehat{x}(\alpha) - widx(\alpha) = 0$, whence $(x - s_F) \perp M_F$. Since $s - s_F \in M_F$ for every $s \in M_F$, we have

$$\|x - s\|^2 = \|x - s_F\|^2 + \|s - s_F\|^2 \tag{$*$}$$

and hence $\|x - s|^2 > |x - s_F\|^2$ unless $\|s - s_F\|^2 = 0$. This gives (1.3.1.4). Setting $s = 0$ in $(*)$, we get $\|x\|^2 \geq \|s_F\|^2$ which is the same as (1.3.1.5). □

1.4. **Sums of infinite non-negative numbers.** Let A be a non-empty set and $s = \{s_a\}_{a \in A}$ a function on A with $s_a \geq 0$ for all $a \in A$. The sum

$$\sum_{a \in A} s_a$$

is defined as the supremum of sums of the form

$$\sum_{a \in F} s_a$$

where F is a finite subset of A. A little thought shows that if $\#(= \#_A)$ is the counting measure on $(A, \mathscr{P}(A))$, then

$$\sum_{a \in A} s_a = \int_A s \, d\#.$$

Lemma 1.4.1. *If* A *and* s *are as above and* $\sum_{a \in A} s_a < \infty$ *then* $\{a \in A \mid s_a \neq 0\}$ *is at most a countable set.*

Proof. Let $M = \sum_{a \in A} s_a$. Let $A_n = \{a \in A \mid s_a \leq 1/n\}$, $n \in \mathbb{N}$. Then

$$\frac{1}{n} \sum_{a \in A_n} 1 \leq \sum_{a \in A_n} s_a \leq M < \infty.$$

It follows that the cardinality of A_n is finite, whence $\operatorname{Supp} s = \cup_{n \in \mathbb{N}} A_n$, is countable. □

1.5. **Isometries.**

Lemma 1.5.1. *Let $f\colon X \to Y$ be continuous map of metric spaces such that X is complete, f is an isometry on a dense subset X_0 of X, and $f(X_0)$ is dense in Y. Then f is a surjective isometry.*

Proof. The fact that f is an isometry follows from trivial considerations. The non-trivial part is the surjectivity of f. To that end, let $y \in Y$. Let $\{y_n\}$ be a sequence in $f(X_0)$ converging to y. Since $f\colon X_0 \to f(X_0)$ is an isometry, there exist unique pre-images $x_n \in X_0$, such that $f(x_n) = y_n$, $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy for $\{y_n\}$, and since X is complete, $\{x_n\}$ converges to a point $x \in X$. By the continuity of f, $y = f(x)$. ☐

1.6. **Orthonormal sets again.**

Theorem 1.6.1. *Let $\{u_\alpha \mid \alpha \in A\}$ be an orthonormal set in H and P the linear span of $\{u_\alpha\}$. The inequality*

$$\sum_{\alpha \in A} |\widehat{x}(\alpha)|^2 \le \|x\|^2$$

holds for all $x \in X$, and $x \mapsto \widehat{x}$ is a continuous linar map from H onto *$\ell^2(A)$ whose restriction to $\overline{P}$, the closure of P in H, is an isometry of $\overline{P}$* onto *$\ell^2(A)$.*

Proof. According to (1.3.1.5) we have

$$\sum_{\alpha \in F} |\widehat{x}(\alpha)|^2 \le \|x\|^2$$

for every finite subset F of A, giving the inequality $\sum_{\alpha \in A} |\widehat{x}(\alpha)|^2 \le \|x\|^2$. It is immediate from this that $x \mapsto \widehat{x}$ is a continuous linear map from H into $\ell^2(A)$. To prove that the map is onto, it is enough to prove the rest of the assertion, namely the restriction of the map to $\overline{P}$ is an isometry onto $\ell^2(A)$.

A simple function $\varphi\colon A \to \mathbb{C}$ such that $\#(\{\varphi \neq 0\}) < \infty$ is the same as a function φ with finite support. We have just proved that the linear space S of functions on A with finite support is dense in $\ell^2(A)$. By Theorem 1.3.1 (a), the map $x \mapsto \widehat{x}$ from $P \to S$ is an isometry onto S. Since $\overline{P}$ is complete (being a closed subspace of H) and P is dense in $\overline{P}$, Lemma 1.5.1 applies and the theorem follows. ☐

Remark 1.6.2. The fact that $x \mapsto \widehat{x}$ maps H onto $\ell^2(A)$ is called the *Riesz-Fischer Theorem*. It has an important history. Many other theorems are also called the Riesz-Fischer theorem, but essentially they can be traced to this fact. We will say more when we do Fourier Series of square integrable periodic functions.

1.7. **Complete orthonormal sets.** We will prove the following theorem in the next class. An orthonormal set satisfying any one of the equivalent conditions of the theorem below is a called a *complete orthonormal set for H* or an *orthonormal basis for H*. Using Zorn's Lemma we will show (next class) that a complete orthonormal set always exists for a Hilbert space. It turns out that two complete orthonormal sets for the same Hilbert space have the same cardinality. But we will not prove that in this course. This cardinality is $\aleph_0$ if (and only if) H is *separable*. Thus all separable Hilbert spaces are isomorphic to ℓ^2.

Theorem 1.7.1. *Let $\{u_\alpha \mid \alpha \in A\}$ be an orthonormal set in H. Then the following conditions are equivalent.*

(a) $\{u_\alpha \mid \alpha \in A\}$ *is a maximal orthonormal set in* H.
(b) *The linear span* P *of* $\{u_\alpha \mid \alpha \in A\}$ *is dense in* H.
(c) *For every* $x \in H$ *we have*

$$\|x\|^2 = \sum_{\alpha \in A} |\widehat{x}(\alpha)|^2.$$

(d) *For every pair of vectors* x, y *in* H *we have*

$$\langle x, y \rangle = \sum_{\alpha \in A} \widehat{x}(\alpha)\overline{\widehat{y}(\alpha)}.$$

References

[R] W. Rudin, *Real and Complex Analysis*, (Third Edition), McGraw-Hill, New York, 1987.