LECTURE 16

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1. Measures on Locally compact Hausdorff spaces (again)

Throughout this section X is a locally compact Hausdorff space.

1.1. The space of continuous functions vanishing at ∞.

Definition 1.1.1. The space $C_0(X)$ is the space of continuous complex functions on X vanishing at infinity with the supremum norm. In other words $C_0(X)$ consists of continuous functions $f\colon X \to \mathbb{C}$ such that

$$K_\epsilon(f) := \{x \in X \mid |f(x)| \geq \epsilon\}$$

is compact for every $\epsilon > 0$, and for $f \in C_0(X)$, the norm of f is

$$\|f\|_\infty := \sup |f|.$$

It is clear that if $\widehat{X} = X \cup \{\infty\}$ is the one point compactification of X, then $C_0(X)$ is precisely the space of continuous functions on $\widehat{X}$ which vanish at ∞, for any continuous complex function f on X such that $K_\epsilon(f)$ is compact for every positive epsilon has a unique extension to $\widehat{X}$ such that the value of the extension at ∞ is zero. Since X is dense in the compact set $\widehat{X}$, the supremum of $|f|$ over X or $\widehat{X}$ is the same.

It is easy to see that $C_0(X)$ is a Banach space with this norm and $C_c(X)$ is a dense subspace of $C_0(X)$ (why and why?).

1.2. Approximation of $L^p(\mu)$ by continuous functions.

Now suppose $\mathfrak{M}$ is a σ-algebra containing $\mathscr{B}(X)$ and μ is a positive measure on $(X, \mathfrak{M})$, satisfying the following conditions:

(a) $\mu(K) < \infty$ for every compact K.
(b) μ is outer regular.
(c) The relation

$$\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\}$$

holds for every open set E and every measurable E with $\mu((E) < \infty$.

Recall that measures arising from positive functionals on $C_c(X)$ satisfy these properties (Riesz Representation for positive functionals on $C_c(X)$). In that case, in addition to (a), (b), and (c), $(\mathfrak{M}, \mu)$ is complete.

Theorem 1.2.1. *For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.*

Proof. Let S be the set of simple measurable functions s such that

$$\mu\big(\{x \in X \mid s(x) \neq 0\}\big) < \infty.$$

Then we know that S is dense in $L^p(\mu)$ (the completeness of $\mathfrak{M}$ is not required for this). Let $s \in S$. Let $\epsilon > 0$ be given. By Lusin's theorem, there exists $g \in C_c(X)$,

$|g| \le |s|_\infty$ such that $g(x) = s(x)$ for every $x \in X$ except on a set of measure $< \epsilon$. Let $E = \{g \ne s\}$. Then

$$\|s - g\|_p^p = \int_E |s - g|^p d\mu \le \int_E (2\|s\|_\infty)^p d\mu \le 2^p \|s\|_\infty^p \epsilon,$$

which means $\|s - g\|_p \le 2\|s\|_\infty \epsilon^{1/p}$. ☐

Remark 1.2.2. Even though Lusin's theorem requires the measure to be complete, a little thought shows that for Theorem 1.2.2 we do not require $(\mathfrak{M}, \mu)$ to be complete. The proof works even without that requirement for S is dense in $L^p(\mu)$ without that requirement and $\{g \ne s\}$ is $\mathfrak{M}$-measurable and hence $\widehat{\mathfrak{M}}$-measurable, where $\widehat{\mathfrak{M}}$ is the completion $\mathfrak{M}$ with respect to μ.

By a regular complex Borel measure we mean a complex measure on $\mathscr{B}(X)$ such that $|\mu|$ is regular.

Theorem 1.2.3. *Let μ be a regular complex Borel measure on X such that*

$$\int_E f d\mu = 0 \qquad (f \in C_0(X)).$$

Then $\mu = 0$.

Proof. Since μ is complex $|\mu|$ is a finite measure. By hypothesis $|\mu|$ is Borel and regular. Let h be a measurable function such that $|h| = 1$ and $d\mu = h d|\mu|$. Then by hypothesis

$$\int_E h f d|\mu| = 0 \qquad (f \in C_0(X)).$$

Since $C_c(X) \subset C_0(X)$, the above gives, for every $f \in C_c(X)$,

$$\begin{aligned} |\mu|(X) &= \int_X h(\bar{h} - f) d|\mu| \\ &\le \int_X |\bar{h} - f| d|\mu|. \end{aligned}$$

Since this is true for every $f \in C_c(X)$, and since by Theorem 1.2.2 the last quantity can be made as small as we wish by a suitable choice of f in $C_c(X)$, we conclude that $|\mu|(X) = 0$. ☐

2. The Riesz Representation Theorem for $C_0(X)^*$

In this section too X is a locally compact Hausdorff space. The point of this section is to show that every bounded linear functional on $C_0(X)$ is of the form $\Phi_\mu = \int_X (-)\, d\mu$ for a unique complex regular Borel measure μ and that the norm of the functional Φ_μ is $|\mu|(X)$.

2.1. **Complex regular Borel measures as bounded functionals.**

Suppose μ is a complex Borel measure on X. Since elements of $C_0(X)$ are bounded, and since $|\mu|(X) < \infty$ we see that $f \in L^1(\mu)$ for every $f \in C_0(X)$. The linear functional

$$\Phi_\mu \colon C_0(X) \longrightarrow \mathbb{C}$$

given by $f \mapsto \int_X f d\mu$ is bounded, for, with $h = d\mu/d|\mu|$ (recall that $|h| = 1$), we have,

$$\left|\int_X f d\mu\right| = \left|\int_X f h\, d|\mu|\right| \le \int_X |f|\, d|\mu| \le \|f\|_\infty |\mu|(X)$$

giving,

$$\|\Phi_\mu\| \le |\mu|(X). \tag{2.1.1}$$

We point out that the proof of (2.1.1) does not use regularity of μ and so is true for any complex measure μ.

Now suppose in addition that μ is regular. If K is a non-empty compact set, we can find $f \in C_c(X)$ such that $K \prec f$. Then,

$$|\mu|(K) \le \int_X f\, d|\mu| = \Phi_\mu(\bar{h}f) \le \|\Phi_\mu\|\|f\|_\infty = \|\Phi_\mu\|.$$

Taking supremum over all compact sets K, and using the fact that $|\mu|$ is regular, we get $|\mu|(X) \le \|\Phi_\mu\|$. This together with (2.1.1) gives

$$\|\Phi_\mu\| = |\mu|(X) \tag{2.1.2}$$

for *regular* complex Borel measures.

2.2. **Positive functionals associated with elements of $C_0(X)^*$.** Let

$$\Phi\colon C_0(X) \to \mathbb{C}$$

be a bounded linear functional. Let $C_c^+(X)$ be the set of $f \in C_c(X)$ such that $f \ge 0$. For $f \in C_c^+(X)$ set

$$\Lambda(f) = \sup\{|\Phi(h)|\colon h \in C_c(X)\}. \tag{2.2.1}$$

Lemma 2.2.2. *Let $f, g \in C_c^+(X)$ and let c be a non-negative real number. Then $\Lambda(f+g) = \Lambda f + \Lambda g$ and $\Lambda cf = c\Lambda f$.*

Proof. The relation $\Lambda cf = c\Lambda f$ is clearly true since Φ is linear. Suppose $h \in C_c(X)$ is such that $|h| \le f+g$. Let V be the set of points x in X such that $f(x)+g(x) > 0$. Then V is an open set and clearly $f(x) = g(x) = 0$ for $x \notin V$. Set

$$h_1(x) = \begin{cases} \dfrac{h(x)f(x)}{f(x)+g(x)} & (x \in V) \\ \\ 0 & (x \notin V) \end{cases} \qquad \text{and} \qquad h_2(x) = \begin{cases} \dfrac{h(x)g(x)}{f(x)+g(x)} & (x \in V) \\ \\ 0 & (x \notin V) \end{cases}$$

Since $|h_1| \le f$ on V and f vanishes on $X \smallsetminus V$, it is clear that $h_1 \in C_c(X)$. Similarly $h_2 \in C_c(X)$. Note that $h_1 + h_2 = h$, $|h_1| \le f$, and $|h_2| \le g$. We therefore have

$$|\Phi h| = |\Phi h_1 + \Phi h_2| \le |\Phi h_1| + |\Phi h_2| \le \Lambda f + \Lambda g.$$

This means

$$\Lambda(f+g) \le \Lambda f + \Lambda g.$$

On the other hand, given $\epsilon > 0$, we can find $h_1, h_2 \in C_c(X)$ such that $|h_1| \le f$, $|h_2| \le g$, and $\Lambda f \le |\Phi h_1| + \epsilon$ and $\Lambda g \le |\Phi h_2| + \epsilon$. Let α_j, $j = 1, 2$, be complex numbers such that $\alpha_j \Phi(h_j) = |\Phi(\alpha_j)|$. Then

$$\Lambda f + \Lambda g \le |\Phi h_1| + |\Phi h_2| + 2\epsilon = \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\epsilon \le \Lambda(f+g) + 2\epsilon.$$

Since ϵ is an arbitrary positive number, we get

$$\Lambda f + \Lambda g \le \Lambda(f+g).$$

The relation $\Lambda(f+g) = \Lambda f + \Lambda g$ follows. ☐

If $f \in C_c(X)$ is real-valued than f^+ and f^- lie in $C_c^+(X)$ and we set define $\Lambda f = \Lambda f^+ - \Lambda f^-$. If f is an arbitrary element of $C_c(X)$, and $f = u + iv$ is the decomposition of f into its real and imaginary parts, define $\Lambda f = \Lambda u + i\Lambda v$. It is quite straightforward to see that

$$\Lambda\colon C_c(X) \longrightarrow \mathbb{C}$$

is linear. By construction it is a positive functional.

Lemma 2.2.3. *Let λ be the positive Borel measure associated to the positive functional Λ on $C_c(X)$. Then Λ is a bounded functional and*

$$\lambda(X) = \|\Lambda\| = \|\Phi\|.$$

In particular λ is a finite measure, and hence regular.

Proof. Using the integral representation of Λ as $\Lambda = \int_X (-)\, d\lambda$, we see that

$$|\Lambda f| \leq \Lambda(|f|).$$

If $f \in C_c(X)$, then for every $h \in C_c(X)$ such that $|h| \leq |f|$ we have

$$|\Phi h| \leq \|\Phi\| \|h\|_\infty \leq \|\Phi\| \|f\|_\infty,$$

giving $\Lambda|f| \leq \|\Phi\| \|f\|_\infty$. This means

$$|\Lambda f| \leq \|\Phi\| \|f\|_\infty,$$

whence Λ is a bounded linear functional and

$$\|\Lambda\| \leq \|\Phi\|.$$

On the other hand, for $f \in C_c(X)$ we have

$$|\Phi(f)| \leq \Lambda|f| \leq \|\Lambda\| \|f\|_\infty.$$

It follows that

$$\|(\Phi|_{C_c(X)})\| \leq \|\Lambda\|.$$

By Hahn-Banach $\Phi|_{C_c(X)}$ can be extended to a bounded linear functional on $C_0(X)$ which preserves norms. However, $C_c(X)$ is dense in $C_0(X)$. It follows that there is only one bounded linear extension of $\Phi|_{C_c(X)}$, namely Φ. Thus $\|\Phi| \leq \|\Lambda\|$ giving

$$\|\Lambda\| = \|\Phi\|.$$

Now, $\lambda(X)$ is (by construction) the supremum of $\{\Lambda f \mid f \prec X\}$. Thus $\lambda(X) \leq \|\Lambda\| = \|\Phi\|$. In particular λ is a finite measure. Now if $f \in C_c(X)$, $|f| \leq 1$, we have

$$|\Lambda f| = \int_X f\, d\lambda \leq \int_X |f|\, d\lambda \leq \int_X 1\, d\lambda = \lambda(X),$$

giving $\|\Lambda\| \leq \lambda(X)$, whence $\Lambda = \lambda(X)$. $\square$

2.3. **The Riesz Representation Theorem for $\mathbf{C}_0(X)^*$.** Here is the main theorem of this section.

Theorem 2.3.1. *Let X be a locally compact Hausdorff space. Then every bounded linear functional Φ on $C_0(X)$ is of the form*

$$\Phi = \int_X (-)\, d\mu$$

for a unique regular complex Borel measure μ and in this case

$$\|\Phi\| = |\mu|(X).$$

Proof. The assertion is that every $\Phi \in C_0(X)^*$ is equal to Φ_μ for a unique (regular, complex, Borel) μ and that $\|\Phi_\mu\| = |\mu|(X)$. The uniqueness assertion is simply Theorem 1.2.3 and the norm assertion is (2.1.2). Thus we only have to prove that $\Phi = \Phi_\mu$ for some regular complex Borel measure μ on X.

Let $\Phi \in \mathbf{C}_0(X)^*$. Let Λ and λ be the associated positive linear functional on $C_c(X)$ and the regular positive Borel measure as in Subsection 2.2. According to Lemma 2.2.3, Λ is bounded and λ is finite and regular. Now as a vector space, $C_c(X)$ is a subspace of both $C_0(X)$ as well of $L^1(\lambda)$. The latter is true because λ is a finite measure and all bounded measurable functions are therefore in $L^1(\lambda)$. Now

$$\Big|\int_X f\,d\lambda\Big| \le \int_X |f|\,d\lambda = \|f\|_{1,\lambda} \qquad (f \in C_c(X)),$$

where $\|\cdot\|_{1,\lambda}$ is the standard norm on $L^1(\lambda)$. Thus $\Phi|_{C_c(X)}$ is a bounded functional on $(C_c(X), \|\cdot\|_{1,\lambda})$ of norm ≤ 1. Let us write Φ' for $\Phi|_{C_c(X)}$ thought of as a bounded linear functional on $(C_c(X), \|\cdot\|_{1,\lambda})$. Now $\|\Phi'\| \le 1$. By the Hahn-Banach Theorem there is a bounded linear extension F of Φ' to $L^1(\lambda)$, such that $\|F\| = \|\Phi'\|$. Thus $\|F\| \le 1$. Now we have seen that $L^1(\lambda)^* = L^\infty(\lambda)$, and hence there exists a unique $g \in L^\infty(\lambda)$ with $\|g\|_\infty \le 1$ such that

$$F(f) = \int_X fg\,d\lambda \qquad (f \in L^1(\lambda)).$$

In particular we have

$$\Phi(f) = \int_X fg\,d\lambda \qquad (f \in C_c(X)).$$

Since $g \in L^\infty(\lambda)$ and λ is a finite measure, $E \mapsto \int_E g\,d\lambda$ defines a complex Borel measure μ on X and in our notation this is expressed as $d\mu = gd\lambda$. The above equation can then be re-written as

$$\Phi|_{C_c(X)} = \Phi_\mu|_{C_c(X)}.$$

Since both Φ and Φ_μ are continuous on $C_0(X)$ and $C_c(X)$ is dense in $C_0(X)$ in the $\|\cdot\|_\infty$ norm, we get

$$(*) \qquad \Phi = \Phi_\mu.$$

It remains to show that μ is regular. From results in previous lectures, we know that $d|\mu| = |g|\,d\lambda$. We therefore have the chain of inequalities

$$|\mu|(X) = \int_X |g|\lambda \ge \int_X |fg|\,d\lambda \ge |\Phi(f)|,$$

for every $f \in C_c(X)$ with $\|f\|_\infty \le 1$. Taking supremums we get

$$(\dagger) \qquad |\mu|(X) \ge \|\Phi\|.$$

On the other hand (2.1.1), which does not require μ to be regular, gives

$$(\ddagger) \qquad |\mu|(X) \le \|\Phi_\mu\|.$$

From $(*)$, $(\dagger)$, and $(\ddagger)$, we get

$$|\mu|(X) = \|\Phi\|.$$

Using Lemma 2.2.3 we get $|\mu|(X) = \lambda(X)$. Now $|g| \le 1$ a.e. $[\lambda]$ and hence $1 - g \ge 0$ a.e. $[\lambda]$. But $\int_X (1-g)\,d\lambda = \lambda(X) - |\mu|(X) = 0$. It follows that $|g| = 1$ a.e. $[\lambda]$. Hence

$$|\mu| = \lambda.$$

This proves that μ is regular. $\square$

Remarks 2.3.2. (i) If X is already compact, then $C_0(X) = C(X)$ and the theorem says that for a compact Hausdorff space $C(X)^*$ is the space of regular complex Borel measures on X with norm given by the total variation of complex measures. The original version of F. Reisz stated that the dual of $C[0,1]$ (with supremum norm) is the space of right continuous functions α of bounded variation on $[0,1]$ with $\alpha(0) = 0$, and with $\|\alpha\|$ given by the total variation of α (please refer to an earlier HW problem for the definitions).

(ii) According to the Theorem on p. 2 of Lecture 14b, if σ is a positive measure on a measurable space and μ is a complex measure on the same space such that $d\mu = g\,d\sigma$ for some $g \in L^1(\sigma)$, then $|\mu|(E) = \int_A |g|\,d\sigma$. In particular

$$(*) \qquad |\mu|(X) = \|g\|_{1,\sigma}.$$

Examples 2.3.3. 1) Suppose A is a non-empty set. For simplicity, assume A is countable. If f is a function on A, it is common to write f_a as well as $f(a)$ for the value of f at $a \in A$, and often the function f is written as $\{f_a\}$ or represented by standard variants like $\{f_a\}_{a\in A}$. The space $\ell^p(A)$ is defined as $L^p(\#)$ where $\#(= \#_A)$ is the counting measure on $(A, \mathscr{P}(A))$. For $p \ge 1$, the symbol $\|f\|_p$ for $f = \{f_a\}\colon A \to \mathbb{C}$ has the usual measure-theoretic meaning for the measure space $(A, \mathscr{P}(A), \#)$, namely $\|f\|_p = \{\int_X |f|^p\}^{1/p} = \{\sum_{a\in A} |f|^p\}^{1/p}$ for $1 \le p < \infty$, and $\|f\|_\infty = \sup_{a\in A}\{|f_a|\}$. Every complex measure μ on $(A, \mathscr{P}(A))$ is absolutely continuous with respect to $\#$ because $\#(S) = 0$ if and only if $S = \emptyset$. The Radon-Nikodym derivative $d\mu/d\#$ is the clearly the function g^μ given by $g^\mu_a = \mu(\{a\})$. Indeed, $\int_E g^\mu\,d\# = \sum_{a\in E} g^\mu_a = \sum_{a\in E}\mu(\{a\}) = \mu(E)$. This shows that $\{g^\mu_a\}$ is in $\ell^1(A)$, since $\mu(E) \in \mathbb{C}$ for every $E \subset A$ and the specific arrangement of elements of E does not affect the sum $\sum_{a\in E} g^\mu_a$. Conversely, given $g = \{g_a\} \in \ell^1(A)$, it is the Radon-Nikodym derivative (with respect to $\#$) of the complex measure μ_g on A given by $d\mu_g = g\,d\#$. Moreover $g^{\mu_g} = g$, and $\mu_{g^\mu} = \mu$. The formula $(*)$ in Remarks 2.3.2 shows that

$$(**) \qquad |\mu|(X) = \|g^\mu\|_1.$$

If $A = \mathbb{N}$, the convention is to write ℓ^p instead of $\ell^p(\mathbb{N})$.

2) In 1) above, there were no topological considerations. Now let $X = \mathbb{N}$ be given the discrete topology and let $\widehat{X} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$. All complex measures on X and $\widehat{X}$ are regular, since every open set in either is clearly σ-compact. Recall (from HW-6) that c is the closed subspace of ℓ^∞ consisting of convergent sequences, and c_0 the closed subspace of ℓ^∞ consisting of sequences converging to zero. Convergent sequences $f = \{f_n\}$ are precisely the functions which can be extended to a continuous function on $\widehat{X}$ with $f_\infty = \lim_{n\to\infty} f_n$. Thus $c = C(\widehat{X})$. Similarly $c_0 = C_0(X)$. We can regard c_0 as the set of elements f in $C(\widehat{X})$ such that $f_\infty = 0$. We point out that $C(\widehat{X}) = C_0(\widehat{X})$ since $\widehat{X}$ is compact.

By the Riesz Representation Theorem for $C_0(X)^*$ and $C_0(\widehat{X})^*$, and the discussion in Example 1), we have $c^* = \ell^1(\mathbb{N} \cup \{\infty\})$ and $c_0^* = \ell^1$ (these are canonical isometric identifications, which is why we have used the equality sign). The bounded functional on c associated to $g \in \ell^1(\mathbb{N} \cap \{\infty\})$ is

$$\{f_n\} \mapsto (\lim_{n\to\infty} f_n)g_\infty + \sum_{n\in\mathbb{N}} f_n g_n.$$

Similarly the bounded functional on c_0 associated to $h \in \ell^1$ is

$$\{f_n\} \mapsto \sum_n f_n h_n.$$

This can be checked by going through the various identifications made.

3) Using the example from 2), since X and $\widehat{X}$ have the same cardinality, the space $\ell^1(\mathbb{N}\cup\{\infty\})$ is isometrically isomorphic to ℓ^1. One such isomorphism is $g \mapsto h$, with $h_1 = g_\infty$ and $h_n = g_{n-1}$, for $n \geq 2$.[1] This identifies c^* with ℓ^1. The natural map $c^* \to c_0^*$ (see HW-6) under these identifications translates to the endomorphism $P\colon \ell^1 \to \ell^1$ given by the "left shift operation" $\{a_n\} \mapsto \{b_n\}$ with $b_n = a_{n+1}$.

3) Continuing with the above, a complex measure $\widehat{\mu}$ on $\widehat{X}$ can clearly be decomposed as

$$\widehat{\mu} = \mu + \alpha\delta_\infty$$

with μ concentrated on X, δ_∞ the Dirac measure at ∞, and $\alpha \in \mathbb{C}$. Conversely any such choice of μ and α gives us a measure $\widehat{\mu}$ on $\widehat{X}$ via the above formula. The measure μ can be regarded as a measure on X. It is easy to see that the natural map $c^* \to c_0^*$ can be identified with $\mu + \alpha\delta_\infty \mapsto \mu$.

References

[R] W. Rudin, *Real and Complex Analysis*, (Third Edition), McGraw-Hill, New York, 1987.

[1] This corresponds to the set theoretic isomorphism $\theta\colon \mathbb{N} \xrightarrow{\sim} \mathbb{N}\cup\{\infty\}$ given by $1 \mapsto \infty$ and $n \mapsto n-1$ for $n \geq 2$, so that $h = g \circ \theta$.