Lemma: If μ is a positive σ-finite measure on $(X, \mathcal{M})$ then there exists a function $w \in L^1(\mu)$ such that $0 < w(x) < 1$ for every $x \in X$.

Lemma: Let μ be σ-finite, $p \in [1, \infty)$, q the conjugate exponent to p, and g an element in $L^q(\mu)$. Let $\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$ be the bounded linear functional $f \longmapsto \int_X fg\, d\mu$ (see Example 2 above). Then $\|\Phi_g\| = \|g\|_q$. In particular, the map $L^q(\mu) \longrightarrow L^p(\mu)^*$ given by $g \longmapsto \Phi_g$ is an injective map.

Proof:

We have already seen that $\|\Phi_g\| \le \|g\|_q$. Suppose $p=1$ (so that $q = \infty$). Pick $f \in L^1(\mu)$ s.t. $0 < f < \infty$ (e.g., the w, $0 < w < 1$, of an earlier Lemma). Then $|\int_E fg\, d\mu| \le \Phi_g(f\chi_E) \le \|\Phi_g\| \int_E f\, d\mu$, $\forall E \in \mathcal{M}$. By the previous Lemma, $\|g\|_\infty \le \|\Phi_g\|$. So the assertion is true when $p=1$.

Suppose $1 < p < \infty$. For some m'ble α, $|\alpha| = 1$, we have $|g| = \alpha g$. Let $f = \alpha |g|^{q-1}$. Then $|f|^p = |g|^{(q-1)p} = |g|^q$, whence $\|f\|_p = \|g\|_q^{q/p}$. Also $\Phi_g(f) = \int_X (\alpha |g|^{q-1}) g\, d\mu = \int_X |g|^q d\mu = \|g\|_q^q$. Thus

$$\|g\|_q^q = \Phi_g(f) \le \|\Phi_g\| \cdot \|f\|_p = \|\Phi_g\| \cdot \|g\|_q^{q/p}, \text{ i.e.}$$

$$\|g\|_q^q \le \|g\|_q^{q/p} \|\Phi_g\|$$

If $\|g\|_q = 0$, there is nothing to prove since in that case $\Phi_g = 0$. Otherwise we have $\|g\|_q^{q - q/p} \le \|\Phi_g\|$, i.e. $\|g\|_q \le \|\Phi_g\|$. q.e.d.

Proposition: Let μ be a finite measure, $1 \le p < \infty$, M a non-negative real constant, and $g \in L^1(\mu)$ such that $|\int_X fg\, d\mu| \le M \cdot \|f\|_p$ for every $f \in L^\infty(\mu)$. Then $\|g\|_q \le M$ where q is the exponent conjugate to p. (In particular $g \in L^q(\mu)$.)

Proof:

Suppose $p=1$. Let $E \in \mathfrak{M}$ and set $f = \chi_E$. Then $f \in L^\infty(\mu)$ and $\|f\|_1 = \mu(E)$. Hence $|\int_E g\, d\mu| = |\int_X fg\, d\mu| \le M \cdot \mu(E)$. By Thm 1.40 Rudin (see Tutorial 2) we get $|g| \le M$ a.e., whence $\|g\|_\infty \le M$ and we are done in this case.

Now suppose $1 < p < \infty$. Let $E_n = \{|g| \le n\}$. We know there exists m'ble α, $|\alpha| = 1$ s.t. $|g| = \alpha g$. Let $f = \alpha |g|^{q-1} \chi_{E_n}$. Then $f \in L^\infty$. Moreover $|f|^p = |g|^q \chi_{E_n}$, whence $\|f\|_p = \left\{ \int_{E_n} |g|^q d\mu \right\}^{\frac{1}{p}}$. Moreover, $fg = |g|^q \chi_{E_n}$. Hence

$$\int_{E_n} |g|^q\, d\mu = \int_X fg\, d\mu \le M \cdot \|f\|_p = M \left\{ \int_{E_n} |g|^q d\mu \right\}^{\frac{1}{p}}.$$

If $g = 0$ a.e., the Proposition is obvious. So assume this is not so. This means $\exists\, N \in \mathbb{N}$ s.t. $\int_{E_n} |g|^q d\mu > 0$ for $n \ge N$. Hence the above inequality gives

$$\left\{ \int_{E_n} |g|^q d\mu \right\}^{\frac{1}{q}} \le M$$

Let $n \longrightarrow \infty$ and use MCT to get the required result. q.e.d

Theorem: Let μ be σ-finite. Let $p \in [1, \infty)$ and let q be the conjugate exponent to p. For $g \in L^q(\mu)$, let $\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$ be the bounded linear functional described above, namely

$$\Phi_g f = \int_X f g\, d\mu \qquad (f \in L^p(\mu)).$$

Then the linear map

$$L^q(\mu) \longrightarrow (L^p(\mu))^*$$

given by $g \longmapsto \Phi_g$ is an isometric isomorphism, i.e. it is one-to-one, onto, and $\|\Phi_g\| = \|g\|_q$ for every $g \in L^q(\mu)$.

Proof:

We have already seen that Φ_g is bounded and $\|\Phi_g\| = \|g\|_q$ for every $g \in L^q(\mu)$. This means that the map $L^q(\mu) \longrightarrow L^p(\mu)^*$ given by $g \longmapsto \Phi_g$, is injective. Our task is to show that for every $\Phi \in L^p(\mu)^*$ there exists a $g \in L^q(\mu)$ such that $\Phi = \Phi_g$.

First assume μ is finite, i.e, $\mu(X) < \infty$. Let $\Phi : L^p(\mu) \longrightarrow \mathbb{C}$ be a bounded linear functional. Since μ is finite $\chi_E \in L^p(\mu)$ for every $E \in \mathcal{M}$. Define

$$\lambda(E) = \Phi(\chi_E).$$

Note that λ is finitely additive, for, if $E = E_1 \cup E_2$, where $E_1 \cap E_2 = \phi$, $E_1, E_2 \in \mathcal{M}$, then $\chi_E = \chi_{E_1} + \chi_{E_2}$. Now suppose $E_1, E_2, \ldots, E_n, \ldots$ are pairwise disjoint m'ble sets with $E = \bigcup_n E_n$. Let $S_n = \bigcup_{k=1}^{n} E_k$ and $T_n = E - S_n$, $n \in \mathbb{N}$. Since $T_n \downarrow \phi$, and μ is finite we have $\lim_{k \to \infty} \mu(T_n) = 0$. Now

$$\|\chi_E - \chi_{S_n}\|_p = \left\{ \int_X (\chi_E - \chi_{S_n})^p \, d\mu \right\}^{1/p}$$

$$= \left\{ \int_X \chi_{T_n}^p \, d\mu \right\}^{1/p}$$

$$= \left\{ \int_X \chi_{T_n} \, d\mu \right\}^{1/p}$$

$$= \mu(T_n)^{1/p} \longrightarrow 0 \text{ as } n \to \infty.$$

Thus $\chi_{S_n} \longrightarrow \chi_E$ in $L^p(\mu)$ as $n \longrightarrow \infty$.

It follows that $\Phi(\chi_{S_n}) \longrightarrow \Phi(\chi_E)$ as $n \to \infty$.

This means $\sum_{n=1}^{\infty} \lambda(E_n) = \lambda(E)$.

Thus λ is a complex measure. If $\mu(E) = 0$, then $\chi_E = 0$ a.e., whence $\lambda(E) = 0$, and hence $\lambda \ll \mu$. By the Radon-Nikodym theorem $\exists\, g \in L^1(\mu)$ s.t.

$$d\lambda = g\, d\mu.$$

Since simple funtions are in $L^p(\mu)$ (once again using the fact that μ is finite), we have for $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ ($A_i \in M$, $i = 1, \dots, n$, $A_i \cap A_j = \phi$, $i \neq j$)

$$\Phi(s) = \sum_{i=1}^{n} a_i\, \Phi(\chi_{A_i}) = \sum_{i=1}^{n} a_i\, \lambda(E_i)$$

$$= \sum_{i=1}^{n} a_i \int_X \chi_{E_i}\, g\, d\mu$$

$$= \int_X s g\, d\mu.$$

Thus $\Phi(s) = \int_X sg\, d\mu$ for every simple m'ble s.

Since μ is finite, $L^\infty(\mu) \subset L^p(\mu)$. Moreover

$$\|f\|_p \leq \|f\|_\infty\, \mu(X)^{1/p} \qquad (f \in L^\infty(\mu))$$

and hence the inclusion map

$$L^\infty(\mu) \hookrightarrow L^p(\mu)$$

is continuous.

Let $f \in L^\infty(\mu)$. Since simple m'ble funtions are dense in $L^\infty(\mu)$, we can find a sequence of simple m'ble funtions $\{s_n\}$ s.t. $\|s_n - f\|_\infty \to 0$ as $n \to \infty$. This means, from what we just said, $\|s_n - f\|_p \to 0$ as $n \to \infty$. Now s_n being simple.

$$\Phi(s_n) = \int_X s_n\, g\, d\mu$$

As $n \longrightarrow \infty$, the LS converges to $\Phi(f)$, since $s_n \longrightarrow f$ in $L^p(\mu)$ and Φ is a bdd functional on $L^p(\mu)$. The R.S. converges to $\int_X fg\, d\mu$ since $s_n \longrightarrow f$ in $L^\infty(\mu)$ and $g \in L^1(\mu)$ (so that $f \longmapsto \int_X fg\, d\mu$ is a bdd functional on $L^\infty(\mu)$).

Thus

$$\Phi(f) = \int_X fg\, d\mu \qquad (f \in L^\infty(\mu)) \qquad (1)$$

Since $\Phi \in L^p(\mu)^*$ and $L^\infty(\mu) \subset L^p(\mu)$ we get

$$\left|\int_X fg\, d\mu\right| = |\Phi(f)| \leq \|\Phi\| \cdot \|f\|_p \qquad (f \in L^\infty(\mu)).$$

From the Proposition we get

$$g \in L^q(\mu) \quad \text{and} \quad \|g\|_q \leq \|\Phi\|.$$

Now both sides of (1) define continuous functionals on $L^p(\mu)$ (the right side because $g \in L^q(\mu)$) and they agree on $L^\infty(\mu)$. Since μ is a finite measure $L^\infty(\mu)$ is dense in $L^p(\mu)$ – in fact simple functions are dense in $L^p(\mu)$. Thus the two sides of (1) agree on $L^p(\mu)$, i.e.,

$$\Phi(f) = \int_X fg\, d\mu \qquad (f \in L^p(\mu)).$$

We are therefore done when μ is a finite measure.

If μ is not finite, since it is σ-finite (by our hypothesis), there exists $w \in L^1(\mu)$, $0 < w < 1$. Let $\tilde{\mu}$ be the measure given by $d\tilde{\mu} = w\, d\mu$. Then $\tilde{\mu}$ is a finite measure. Moreover, for every $r \in [1, \infty)$, $f \longmapsto w^{-1/r} f$ gives an isometric isomorphism between $L^r(\mu)$ and $L^r(\tilde{w})$. This works for $r = \infty$ too if we set $\frac{1}{\infty} = 0$.

So suppose $\Phi \in L^p(\mu)^*$. Define $\tilde{\Phi} : L^p(\tilde{\mu}) \longrightarrow \mathbb{C}$ by

$\tilde{\Phi}(F) = \Phi(w^{1/p} F)$. Then $\|\tilde{\Phi}\| = \|\Phi\|$. Since $\tilde{\mu}$ is finite, we have a unique $G \in L^q(\tilde{\mu})$ such that $\tilde{\Phi}(F) = \int_X FG\,d\tilde{\mu}$ and $\|G\|_{q,\tilde{\mu}} = \|\tilde{\Phi}\| = \|\Phi\|$. If $p=1$, set $g = G$. If $1<p<\infty$ set $g = w^{1/q} G$. Then $g \in L^q(\mu)$ and $\|g\|_q = \|G\|_{q,\tilde{\mu}} = \|\Phi\|$. Moreover for $f \in L^p(\mu)$

$$\begin{aligned}
\Phi(f) = \tilde{\Phi}(w^{-1/p} f) &= \int_X w^{-1/p} f \, G\,d\tilde{\mu} \\
&= \int_X w^{-1/p} f \, w^{-1/q} g \, d\tilde{\mu} \quad (w^{-1/q}=1, \text{ if } q=\infty). \\
&= \int_X w^{(-\frac{1}{p}-\frac{1}{q})} fg\,d\tilde{\mu} \\
&= \int_X fg\, w^{-1}\,d\tilde{\mu} \\
&= \int_X fg\,d\mu \\
&= \Phi_g(f).
\end{aligned}$$

Thus $\Phi = \Phi_g$. We have already seen $\|\Phi\| = \|g\|_q$. q.e.d.