October 4, 2018

Lecture 15

Throughout $(X, \mathcal{M}, \mu)$ is a m'ble space with μ a positive measure.

Dual spaces of $L^p(\mu)$

Recall (from HW assignments, quizzes, earlier lectures, tutorials) that if N is a normed linear space then a bounded linear functional $\Phi: N \longrightarrow \mathbb{C}$ is a linear functional for which there is a constant $M \geq 0$ such that

$$|\Phi x| \leq M \cdot \|x\|. \qquad (x \in N)$$

The norm of Φ, written $\|\Phi\|$, it the smallest M such that the above condition holds. Thus if we have an M as above, then by defn, $\|\Phi\| \leq M$.

It is well-known that bounded linear functionals on N are precisely the continuous linear funtionals on N. The space of bounded linear funtionals is denoted on N is denoted N^* and with the above norm is a Banach space (even if N is not!). N^* is called the dual of N.

There are other well-known descriptions of the above norm on N^*. In fact

$$\|\Phi\| = \sup_{\|x\|=1} |\Phi(x)| = \sup_{\|x\| \leq 1} |\Phi(x)|$$

Examples

1. As we saw in the mid-term exam, $(\ell^1)^* = \ell^\infty$.
2. Let $1 \leq p \leq \infty$. Let q be the conjugate exponent (so that

$\frac{1}{p}+\frac{1}{q}=1$, with the understanding that if $p=1$, $q=\infty$ and if $p=\infty$ then $q=1$. We have seen that if $f\in L^p(\mu)$ and $g\in L^q(\mu)$ then $fg\in L^1(\mu)$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q \qquad \text{———} (*)$$

If $g\in L^q(\mu)$ is fixed and

$$\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$$

is the linear functional

$$\Phi_g(f) = \int_X fg\, d\mu \qquad (f\in L^1(\mu))$$

then by $(*)$ Φ_g is a bounded linear functional and

$$\|\Phi_g\| \le \|g\|_q .$$

In fact equality holds as we'll see soon.

Recall that if $f\ge 0$ is m'ble we can find simple functions s_n, $n\in\mathbb{N}$, $0\le s_n\le f$ such that $s_n\longrightarrow f$ as $n\to\infty$. The specific construction (see Lecture 3) was as follows. First let

$$k_{n,m} = \frac{m}{2^n}\chi_{\left[\frac{m}{2^n},\frac{m+1}{2^n}\right)} \qquad \begin{array}{l} m=0,1,2,\dots \\ n=1,2,\dots \end{array}$$

Define $\varphi_n : [0,\infty] \longrightarrow [0,\infty)$ by

$$\varphi_n = \sum_{m=0}^{2^n\cdot n-1} k_{n,m} + n\chi_{[n,\infty]} \qquad \left(\frac{m+1}{2^n}=n \iff m=n2^n-1\right)$$

Then $0\le\varphi_n(t)\le t$ $\forall\, t\in[0,\infty]$, $\varphi_n<\infty$, $n\in\mathbb{N}$ and $\varphi_n(t)\uparrow t$ as $n\longrightarrow\infty$. If we set $s_n=\varphi_n\circ f$, then $\{s_n\}$ has all the properties we require ($0\le s_n\le f_n$, s_n m'ble, $s_n<\infty$, $s_n\uparrow f$ as $n\longrightarrow\infty$).

If $I = [a,b]$, $0 \le a < b < \infty$, and id_I is the identity function on I (i.e. $id_I(t) = t$, $t \in I$) then clearly $\varphi_n \longrightarrow id_I$ uniformly on I. This means if $f : X \to [0,\infty]$ is bounded then and $s_n = \varphi_n \circ f$, then $s_n \longrightarrow f$ uniformly. We have thus proved (by breaking up f as $f = f^+ - f^-$, and applying the above argument to f^+ and f^-):

$\|\varphi_n - id_I\|_\infty \le \frac{1}{2^n} \quad \forall n \ge b.$

Theorem: The set of measurable simple functions on X is dense in $L^\infty(\mu)$.

We have already seen that the set S of simple functions s on X such that $\mu(\{s \neq 0\}) < \infty$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

Lemma: Suppose g is a complex m'ble function, $f \in L^1(\mu)$, $0 < f(x) < \infty$, $\forall\, x \in X$, and $M > 0$ a constant, and suppose $gf \in L^1(\mu)$ and

$$\left| \int_E gf\, d\mu \right| \le M \int_E f\, d\mu \qquad (E \in \mathfrak{M}).$$

Then $g \in L^\infty(\mu)$ and $\|g\|_\infty \le M$.

Proof:

Let σ be the measure given by $d\sigma = f\, d\mu$. Since $f \in L^1(\mu)$, and $f > 0$, σ is finite measure. Moreover σ is equivalent to μ, i.e., $\sigma \ll \mu$ and $\mu \ll \sigma$, for $d\mu = \frac{1}{f}\, d\sigma$. Now the given condition is equivalent to:

$$\frac{\left| \int_E g\, d\sigma \right|}{\sigma(E)} \le M \qquad (E \in \mathfrak{M} \text{ s.t. } \sigma(E) > 0).$$

This means $|g| \le M$ a.e. $[\sigma]$ (Thm 1.40 Rudin). Since $\mu \ll \sigma$, we are done. q.e.d.

Lemma: Let μ be σ-finite, $p \in [1,\infty)$, q the conjugate exponent to p, and g an element in $L^q(\mu)$. Let $\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$ be the bounded linear functional $f \mapsto \int_X fg\,d\mu$ (see Example 2 above). Then $\|\Phi_g\| = \|g\|_q$. In particular, the map $L^q(\mu) \longrightarrow L^p(\mu)^*$, $g \mapsto \Phi_g$, is injective.

Proof:

We have already seen that $\|\Phi_g\| \le \|g\|_q$. Suppose $p=1$ (so that $q=\infty$). Pick $f \in L^1(\mu)$ s.t. $0<f<\infty$ (e.g., the w, $0<w<1$, of an earlier Lemma). Then $\left|\int_E fg\,d\mu\right| \le \Phi_g(f\chi_E) \le \|\Phi_g\| \int_E f\,d\mu$, $\forall E \in \mathcal{M}$. By the previous Lemma, $\|g\|_\infty \le \|\Phi_g\|$. So the assertion is true when $p=1$.

Suppose $1<p<\infty$. For some m'ble α, $|\alpha|=1$, we have $|g| = \alpha g$. Let $f = \alpha |g|^{q-1}$. Then $|f|^p = |g|^{(q-1)p} = |g|^q$, whence $\|f\|_p = \|g\|_q^{q/p}$. Also $\Phi_g(f) = \int_X (\alpha |g|^{q-1})\, g\,d\mu = \int_X |g|^q d\mu = \|g\|_q^q$. Thus

$$\|g\|_q^q = \Phi_g(f) \le \|\Phi_g\| \cdot \|f\|_p = \|\Phi_g\| \cdot \|g\|_q^{q/p}, \text{ i.e.}$$

$$\|g\|_q^q \le \|g\|_q^{q/p} \|\Phi_g\|$$

If $\|g\|_q = 0$, there is nothing to prove since in that case $\Phi_g = 0$. Otherwise we have $\|g\|_q^{q-q/p} \le \|\Phi_g\|$, i.e. $\|g\|_q \le \|\Phi_g\|$. q.e.d.

Another way of converting σ-finite situations to finite situations:

Suppose μ is positive and σ-finite on $(X, \mathcal{M})$. Let $X = \bigcup_{n=1}^\infty E_n$, $E_n \in \mathcal{M}$, $n \in \mathbb{N}$, with $\mu(E_n) < \infty$ $\forall n \in \mathbb{N}$. Let

$$w_n = \frac{1}{2^n (1+\mu(E_n))} \chi_{E_n}.$$

Set $w = \sum_n w_n$. Then $w \in L^1(\mu)$ and $0<w<1$, as is easy to verify. We thus have: