Complex measures as indefinite integrals of positive measures:

Let μ be a complex measure on $(X, \mathcal{M})$. Clearly $\mu \ll |\mu|$. Hence $\exists\, h \in L^1(|\mu|)$ such that

$$d\mu = h\, d|\mu| \longleftarrow \text{Recall this is short hand for } ``\mu(E) = \int_E h\, d|\mu| \quad \forall E \in \mathcal{M}"$$

Let $E \in \mathcal{M}$ be s.t. $|\mu|(E) > 0$. Then

$$\left| \frac{1}{|\mu|(E)} \int_E h\, d|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1,$$

i.e. $\frac{1}{|\mu|(E)} \int_E h\, d|\mu| \in \bar{\Delta}$, where $\bar{\Delta}$ is the unit disc in $\mathbb{C}$, i.e. $\bar{\Delta} = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$. Since $|\mu|$ is a finite measure according to the results in Tutorial 2 (posted on moodle) or Thm 1.40 of Rudin, h takes values in $\bar{\Delta}$ a.e $[|\mu|]$. This gives

$$|h| \leq 1 \quad \text{a.e. } [|\mu|] \text{ ——————— } (*)$$

We claim $|h| = 1$ a.e.. To see this consider $A_r = \{ |h| < r \}$ for $r > 0$. Let $\{E_j\}$ be a countable m'ble partition of A_r. Then

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} h\, d|\mu| \right| \leq \sum_j \int_{E_j} |h|\, d|\mu|$$

$$= \int_{A_r} |h|\, d|\mu|$$

$$\leq r\, |\mu|(A_r).$$

This means $|\mu|(A_r) \leq r\, |\mu|(A_r)$, i.e.,

$$(r-1)\, |\mu|(A_r) \geq 0.$$

If $r < 1$, the above implies $|\mu|(A_r) = 0$, which in turn means $|h| \geq r$ a.e. $[|\mu|]$ for every $r < 1$. Thus $|h| \geq 1$ a.e. $[\mu]$. In view of $(*)$ above this means $|h| = 1$ (after modifying on $|\mu|$-null set). We thus have:

Theorem: Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ in X. Then there exists a m'ble function h s.t. $|h(x)|=1$ for all $x \in X$ and s.t.

$$d\mu = h\, d|\mu|.$$

Proof:

We already know that $h = \frac{d|\mu|}{d\mu}$ has absolute value 1 a.e. $[|\mu|]$. Let $B = \{|h| \neq 1\}$. Then $|\mu|(B) = 0$. Redefine h so that $h(x) = 1$ for $x \in B$. q.e.d.

Remark: The statement is true even when μ is a σ-finite extended signed measure as is seen by breaking up X into a countable disjoint union of m'ble sets on each of which $|\mu|$ is finite.

Theorem: Suppose μ is a positive measure on $\mathfrak{M}$, $g \in L^1(\mu)$ and

$$\lambda(E) = \int_E g\, d\mu \qquad (E \in \mathfrak{M}).$$

Then

$$|\lambda|(E) = \int_E |g|\, d\mu. \qquad (E \in \mathfrak{M}).$$

Remark: We are not assuming μ is σ-finite!!

Proof:

From the previous theorem, $\exists\, h \in L^1(|\lambda|)$ such that $|h| = 1$ and $d\lambda = h\, d|\lambda|$. In other words

$$\int_E h\, d|\lambda| = \int_E g\, d\mu \qquad (E \in \mathfrak{M}).$$

This means $\int_X s h\, d|\lambda| = \int_X s g\, d\mu$ for every m'ble simple function s on X. Since every bounded m'ble funtion is argument given later.

the uniform limit of simple functions this gives, by DCT, that

$\int_X f\, h\, d|\lambda| = \int_X f g\, d\mu$ for every bdd m'ble function f.

Take $f = \bar{h}\chi_E$, where $E \in \mathcal{M}$. Get

$$\int_E d|\lambda| = \int_E \bar{h} g\, d\mu \qquad (E \in \mathcal{M}).$$

Thus $\int_E \bar{h} g\, d\mu = |\lambda|(E)\ \forall E \in \mathcal{M}$. This means $\bar{h}g$ is non-negative valued a.e. $[\mu]$. Hence $\bar{h}g = |\bar{h}g| = |g|$ a.e. $[\mu]$.

This in turn gives

$$|\lambda|(E) = \int_E |g|\, d\mu \qquad (E \in \mathcal{M}).$$ q.e.d.

Same as saying $|\mu|$ is σ-finite

The Hahn Decomposition Theorem: Suppose μ is a σ-finite extended signed measure. Then there exist sets A and B in $\mathcal{M}$ such that $A \cup B = X$, $A \cap B = \phi$ and such that

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = \mu(B \cap E) \qquad (E \in \mathcal{M}).$$

Proof:

Let $h = \frac{d|\mu|}{d\mu}$, with $|h| = 1$ everywhere. Since h is real (for $|\mu|$ and μ are), h only takes values $+1$ and -1. Let $A = \{h = 1\}$ and $B = \{h = -1\}$. It is clear that the the theorem is true with this A and B. q.e.d.

Corollary (Minimality of the Jordan decomposition): Suppose $\mu = \lambda_1 - \lambda_2$ where λ_1 and λ_2 are positive measures. Then $\mu^+ \leq \lambda_1$ and $\mu^- \leq \lambda_2$.

Proof:

Since $\mu \leq \lambda_1$, we have

$$\mu^+(E) = \mu(E \cap A) \leq \lambda_1(E \cap A) \leq \lambda_1(E).$$ q.e.d.

Remark: The minimality of the Jordan decomposition shown in the last lecture has another consequence. Suppose, as in the Theorem, $d\lambda = g\,d\mu$ where $g \in L^1(\mu)$ (g being "locally $L^1(\mu)$" enough actually), and say g is real-valued, so that λ is a signed measure. Let σ and τ be the positive measures given by $d\sigma = g^+ d\mu$ and $d\tau = g^- d\mu$. Then $\lambda = \sigma - \tau$, whence $\sigma \geq \lambda^+$ and $\tau \geq \lambda^-$.

Let $g_1 = \frac{d\lambda^+}{d\mu}$ and $g_2 = \frac{d\lambda^-}{d\mu}$. Then $\sigma \geq \lambda^+ \Rightarrow \int_E g^+ d\mu \geq \int_E g_1\, d\mu$ $\forall\, E \in \mathcal{M}$, whence $g^+ \geq g_1$ a.e. $[\mu]$. Similarly $g^- \geq g_2$ a.e. $[\mu]$. However $g^+ - g^- = g = g_1 - g_2$ whence, by the minimality of the decomposition $g = g^+ - g^-$ we have $g^+ = g_1$, $g^- = g_2$ a.e. $[\mu]$. After redefining on a μ-null set, we get

$$g^+ = \frac{d\lambda^+}{d\mu} \quad \text{and} \quad g^- = \frac{d\lambda^-}{d\mu}.$$

There are other ways of proving this, which are essentially similar. For example, since $|g|\,d\mu = d|\lambda|$, we have $g_1 + g_2 = |g| = g^+ + g^-$, a.e. $[\mu]$. This coupled with $g_1 - g_2 = g = g^+ - g^-$ a.e. $[\mu]$ forces $g_1 = g^+$ and $g_2 = g^-$ a.e. $[\mu]$.

Theorem (Lebesgue-Radon-Nikodym for σ-finite measures) Suppose μ is a σ-finite positive measure on $(X, \mathcal{M})$ and λ a σ-finite extended signed measure on $(X, \mathcal{M})$.

(a) There is a unique pair of complex measures λ_a and λ_s on $\mathcal{M}$ such that

$$\lambda = \lambda_a + \lambda_s, \quad \lambda_a \ll \mu, \quad \lambda_s \perp \mu.$$

(b) There is a measurable function $h : X \longrightarrow [-\infty, \infty]$ such that one of $\int_X h^+ d\mu$ or $\int_X h^- d\mu$ is finite and

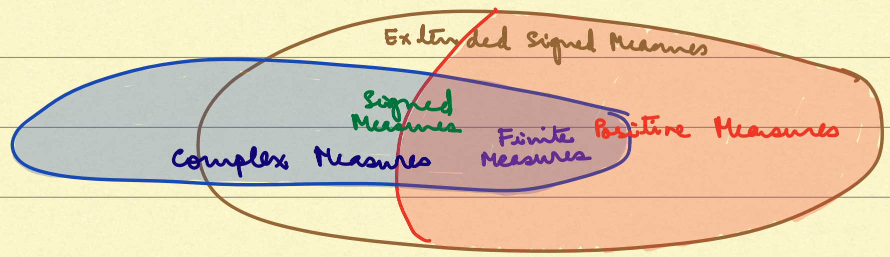
$$\lambda(E) = \int_E h \, d\mu .$$

Moreover $d\lambda^+ = h^+ d\mu$, $d\lambda^- = h^- d\mu$ and $d|\lambda| = |h| d\mu$.

Proof:

The proof is clear from the remark above. The σ-finiteness of λ (i.e., of $|\lambda|$) is used to break up X as $X = \bigcup_{n=1}^{\infty} E_n$, $|\lambda|(E_n) < \infty$, $n \in \mathbb{N}$, with E_n's being pairwise disjoint. The usual Lebesgue-Radon-Nikodym applies to $(\lambda|_{E_n}, \mu|_{E_n})$ giving the result. q.e.d.

Remark: Here is the relationship between complex, signed, extended signed, finite, and positive measures.



$$\begin{array}{ccccc} CM & \supset & SM & \supset & FM \\ & & \cap & & \cap \\ & & ESM & \supset & PM \end{array}$$

Signed measures = Real measures

$$ESM \cap CM = SM$$

$$SM \cap PM = CM \cap PM = FM$$

Integration with respect to a complex measure:

Let μ be a complex measure on $(X, \mathfrak{M})$. A m'ble function $f : X \longrightarrow \mathbb{C}$ is said to be in $L^1(\mu)$ if it is in $L^1(|\mu|)$ and in this case, with $h = d\mu/d|\mu|$, we set

$$\int_X f \, d\mu = \int_X fh \, d|\mu|.$$

Since $|fh| = |f|$ and $f \in L^1(|\mu|)$ the R.S. is a well-defined complex number.

With this definition the relation

$$\mu(E) = \int_X \chi_E \, d\mu \qquad (E \in \mathfrak{M})$$

holds. Moreover, using this definition, it is easy to see that

$$\int_X f \chi_E \, d\mu = \int_E (f|_E) \, d(\mu|_E) \qquad (E \in \mathfrak{M}).$$

The right side is written in a less cumbersome way as $\int_E f \, d\mu$.