October 2, 2018 ← Two lectures given on this day.

Lecture 14

Throughout $(X, \mathcal{M})$ is a m'ble space. Measures are on this space. If λ is an extended signed measure on $\mathcal{M}$ we will make the tacit assumption that $\lambda(E) \neq -\infty$ for any $E \in \mathcal{M}$. By replacing λ with $-\lambda$ if necessary, we can always achieve this.

Absolute continuity: Let μ be a positive measure on $\mathcal{M}$ and λ an arbitrary measure (i.e., either complex or extended signed) on $\mathcal{M}$. The notion of absolute continuity and mutual singularity of measures extends to our more general situation in an obvious way. For completeness we give the formal definition. We say λ is *absolutely continuous* with respect to μ, and write

$$\lambda \ll \mu$$

if $\lambda(E)=0$ for every $E \in \mathcal{M}$ for which $\mu(E)=0$.

Remark: Since μ is a positive measure, if $\mu(E)=0$ for some $E \in \mathcal{M}$, then $\mu(S)=0$ $\forall$ $S \in \mathcal{M}$ s.t. $S \subseteq E$. Suppose

$$\lambda \ll \mu.$$

In view of what we just observed, we see that $\lambda(S)=0$ for every m'ble subset S of E, whence $|\lambda|(E)=0$. If λ is an extended signed measure this yields that λ^+ and λ^- are also absolutely continuous w.r.t. μ. If λ is a complex measure, say $\lambda = \lambda_1 + i\lambda_2$ with λ_1, λ_2 signed measures, then clearly $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, and it follows that $|\lambda_k|, \lambda_k^+, \lambda_k^-$ are all absolutely cts w.r.t μ for $k=1,2$.

In summary if $\lambda \ll \mu$ then:

1. In case λ is a complex measure, λ_1, λ_2 its real and imaginary parts then $|\lambda|, |\lambda_k|, \lambda_k^+, \lambda_k^-$, $k=1,2$ are all absolutely continuous with respect to μ.
2. If λ is an extended signed measure then, $|\lambda|, \lambda^+, \lambda^-$ are all absolutely continuous with respect to μ.

Measures singular to each other: If μ is an arbitrary measure (complex or extended signed) the notion of μ being concentrated in a m'ble set A is defined the same way as it was for the case when μ was a positive measure (see Quiz 3). Similarly, if μ and λ are arbitrary measures we can define what is means for μ and λ to be mutually singular $(\mu \perp \lambda)$.

Recall that in Quiz 3 you were asked to show (for positive measures λ, ν and μ) the following:

$$(*)\quad \begin{cases} \text{(i)} & \nu \perp \mu \text{ and } \lambda \perp \mu \Rightarrow \nu + \lambda \perp \mu. \\ \text{(ii)} & \nu \perp \mu \text{ and } \lambda \ll \mu \Rightarrow \nu \perp \lambda. \\ \text{(iii)} & \nu \perp \lambda \text{ and } \nu \ll \lambda \Rightarrow \nu = 0. \end{cases}$$

These assertions remain true in the present extended case too. More generally, it is not hard to show

Proposition: Suppose $\mu, \lambda, \lambda_1, \lambda_2$ are (arbitrary) measures on $\mathcal{M}$, and μ is positive.

(a) If λ is concentrated on A, so is $|\lambda|$.

(b) If $\lambda_1 \perp \lambda_2$ then $|\lambda_1| \perp |\lambda_2|$

(c) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.

(d) $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.

(e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

(f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$

(g) If $\lambda \ll \mu$ and $\lambda \perp \mu$ then $\lambda = 0$.

Proof:

The positive measure versions of (c), (f) and (g) are (i), (ii), and (iii) of (*) and the proofs are identical. We proved (e) earlier in this lecture. The proof of (a) and (b) are very similar to that. Assertion (d) is easily verified to be true. //

Definition: An extended signed measure λ is σ-finite on $(X, \mathcal{M})$ if X is the countable union of sets $E_n \in \mathcal{M}$ with $\lambda|_{E_n}$ a signed measure for every n. (This is equivalent to requiring that $|\lambda|(E_n) < \infty$ for every n.)

- Note that the notion agrees with the notion of σ-finite for a positive measure.
- The sets E_n can be taken to be pairwise disjoint (set $E_1' = E_1$, $E_2' = E_2 \setminus E_1$, $E_3' = E_3 \setminus (E_1 \cup E_2)$ ....). Or at the other extreme they can be taken to be an increasing sequence $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$

We are now in a position to prove the most important theorem in measure theory, the Radon-Nikodym Theorem, or its more complete version — the Lebesgue-Radon-Nikodym decomposition theorem.

Theorem (The Theorem of Lebesgue-Radon-Nikodym):

Let μ be a positive σ-finite measure on $(X, \mathcal{M})$ and λ a complex measure on $\mathcal{M}$.

(a) There is a unique pair of complex measures λ_a and λ_s on $\mathcal{M}$ such that

$$\lambda = \lambda_a + \lambda_s, \quad \lambda_a \ll \mu, \quad \lambda_s \perp \mu.$$

If λ is a signed then so are λ_a and λ_s. If λ is finite (i.e, λ is positive, with $\lambda(X) < \infty$) then so are λ_a and λ_s.

(b) There is a unique $h \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h \, d\mu$$

for every $E \in \mathcal{M}$.

Proof:

Uniqueness is clear from part (g) of the Proposition above. By breaking up X into a countable disjoint union of m'ble sets on each of which μ is finite, we are reduced to the case where μ is finite. By decomposing λ into its real and imaginary parts, we are reduced to the case where λ is real (i.e., signed). By decomposing the signed measure λ into its positive and negative variations, we are reduced to the case where λ is a positive finite measure. Thus we are now in a situation where μ and λ are both positive finite measures. This case was done in the midterm exam (see solutions in moodle site). Moreover, in this case λ_a and λ_s were finite positive measures. Working backwards, the assertions in (a) when λ is complex or signed follow.

q.e.d.

Remark : The Lebesgue-Radon-Nikodym theorem remains true even if λ is taken to be a σ-finite extended signed measure. In this case λ_a and λ_s are extended signed measures which are σ-finite. The function h in (b) is "locally in L^1", i.e. $h|_E$ is in $L^1(\mu|_E)$ for every $E \in \mathcal{M}$ s.t. $|\lambda|(E) < \infty$. This can be seen by writing $X = \bigcup_{n=1}^{\infty} E_n$, $E_n \in \mathcal{M}$, $E_n \cap E_m = \phi$, if $n \neq m$, $n, m \in \mathbb{N}$. Then the assertions follow. We will see later that $\sigma^+(E) = \int_E h^+ d\mu$, $\sigma^-(E) = \int_E h^- d\mu$, and at least one of h^+ or h^- is in $L^1(\mu)$.

Remark: The most important way the L-R-N thm is used is the following. Suppose μ is positive and σ-finite and λ is complex with $\lambda \ll \mu$. Then $\exists!\, h \in L^1(\mu)$ such that

(*) $$\lambda(E) = \int_E h\, d\mu. \qquad (E \in \mathcal{M})$$

Sometimes it is also used when σ is positive and σ-finite, in which case we have $h \geq 0$ m'ble satisfying (*) above. It can of course be used more generally when λ is extended signed and σ-finite with the caveats about h being "locally in L^1" mentioned above. This is the Radon-Nikodym theorem.

The function h is called the Radon-Nikodym derivative of λ with respect to μ and we write

$$\boxed{h = \frac{d\lambda}{d\mu}}$$

As in Lecture 4, it is not hard to see that if λ is positive (in addition to other hypotheses) and $\nu \ll \lambda$, then $\nu \ll \mu$ and

$$d\nu/d\mu = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu}.$$ (See results in p3 of Lecture 4b to conclude $\int_E \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu} \cdot d\mu = \nu(E)$.)

Another way of converting σ-finite situations to finite situations:

Suppose μ is positive and σ-finite on $(X, \mathcal{M})$. Let $X = \bigcup_{n=1}^{\infty} E_n$, $E_n \in \mathcal{M}$, $n \in \mathbb{N}$, with $\mu(E_n) < \infty$ $\forall n \in \mathbb{N}$. Let

$$w_n = \frac{1}{2^n (1 + \mu(E_n))} \chi_{E_n}.$$

Set $w = \sum_1 w_n$. Then $w \in L^1(\mu)$ and $0 < w < 1$, as is easy to verify. We thus have:

Lemma: If μ is a positive σ-finite measure on $(X, \mathcal{M})$ then there exists a function $w \in L^1(\mu)$ such that $0 < w(x) < 1$ for every $x \in X$.

The proof has already been supplied.

It follows that $\tilde{\mu} : \mathcal{M} \longrightarrow \mathbb{R}$ given by

$$\tilde{\mu}(E) = \int_E w \, d\mu \quad , \quad E \in \mathcal{M}$$

is a finite measure and since $w > 0$, we have by Problem 9(b) of AW5,

$$\mu(E) = \int_E \frac{1}{w} \, d\tilde{\mu}.$$

In fact Problem 9(b) of AW5 together with the Radon-Nikodym statement can be re-phrased as follows: Suppose λ and μ are σ-finite positive measures with $\lambda \ll \mu$ and $\mu \ll \lambda$. Then $0 < \frac{d\lambda}{d\mu} < \infty$ and

$$\boxed{\frac{d\mu}{d\lambda} = \left(\frac{d\lambda}{d\mu} \right)^{-1}}$$

Complex measures as indefinite integrals of positive measures :

Let μ be a complex measure on $(X, \mathfrak{M})$. Clearly $\mu \ll |\mu|$. Hence $\exists\ h \in L^1(|\mu|)$ such that

$$d\mu = h\, d|\mu| \longleftarrow \text{Recall this is short hand for "}\mu(E) = \int_E h\, d|\mu| \ \forall E \in \mathfrak{M}\text{"}$$

Let $E \in \mathfrak{M}$ be s.t. $|\mu|(E) > 0$. Then

$$\left| \frac{1}{|\mu|(E)} \int_E h\, d|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1,$$

i.e. $\frac{1}{|\mu|(E)} \int_E h\, d|\mu| \in \bar{\Delta}$, where $\bar{\Delta}$ is the unit disc in $\mathbb{C}$, i.e. $\bar{\Delta} = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$. Since $|\mu|$ is a finite measure according to the results in Tutorial 2 (posted on moodle) or Thm 1.40 of Rudin, h takes values in $\bar{\Delta}$ a.e $[|\mu|]$. This gives

$$|h| \leq 1 \quad \text{a.e. } [|\mu|] \text{ ———————— } (*)$$

We claim $|h| = 1$ a.e.. To see this consider $A_r = \{ |h| < r \}$ for $r > 0$. Let $\{E_j\}$ be a countable m'ble partition of A_r. Then

$$\begin{aligned}
\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} h\, d|\mu| \right| &\leq \sum_j \int_{E_j} |h|\, d|\mu| \\
&= \int_{A_r} |h|\, d|\mu| \\
&\leq r\, |\mu|(A_r).
\end{aligned}$$

This means $|\mu|(A_r) \leq r\, |\mu|(A_r)$, i.e.,

$$(r-1)\, |\mu|(A_r) \geq 0.$$

If $r < 1$, the above implies $|\mu|(A_r) = 0$, which in turn means $|h| \geq r$ a.e. $[|\mu|]$ for every $r < 1$. Thus $|h| \geq 1$ a.e. $[\mu]$. In view of $(*)$ above this means $|h| = 1$ (after modifying on $|\mu|$-null set). We thus have :

Theorem: Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ in X. Then there exists a m'ble function h s.t. $|h(x)| = 1$ for all $x \in X$ and s.t.

$$d\mu = h\, d|\mu|.$$

Proof:

We already know that $h = \frac{d|\mu|}{d\mu}$ has absolute value 1 a.e. $[|\mu|]$. Let $B = \{|h| \neq 1\}$. Then $|\mu|(B) = 0$. Redefine h so that $h(x) = 1$ for $x \in B$. q.e.d.

Remark: The statement is true even when μ is a σ-finite extended signed measure as is seen by breaking up X into a countable disjoint union of m'ble sets on each of which $|\mu|$ is finite.

Theorem: Suppose μ is a positive measure on $\mathfrak{M}$, $g \in L^1(\mu)$ and

$$\lambda(E) = \int_E g\, d\mu \qquad (E \in \mathfrak{M}).$$

Then

$$|\lambda|(E) = \int_E |g|\, d\mu. \qquad (E \in \mathfrak{M}).$$

Remark: We are not assuming μ is σ-finite!!

Proof:

From the previous theorem, $\exists\, h \in L^1(|\lambda|)$ such that $|h| = 1$ and $d\lambda = h\, d|\lambda|$. In other words

$$\int_E h\, d|\lambda| = \int_E g\, d\mu \qquad (E \in \mathfrak{M}).$$

This means $\int_X s h\, d|\lambda| = \int_X s g\, d\mu$ for every m'ble simple function s on X. Since every bounded m'ble function is argument given later.

the uniform limit of simple functions this gives, by DCT, that

$\int_X f\, h\, d|\lambda| = \int_X fg\, d\mu$ for every bdd m'ble function f.

Take $f = \bar{h}\, \chi_E$, where $E \in \mathcal{M}$. Get

$$\int_E d|\lambda| = \int_E \bar{h} g\, d\mu \qquad (E \in \mathcal{M}).$$

Thus $\int_E \bar{h} g\, d\mu = |\lambda|(E)\ \forall E \in \mathcal{M}$. This means $\bar{h}g$ is non-negative valued a.e. $[\mu]$. Hence $\bar{h}g = |\bar{h}g| = |g|$ a.e. $[\mu]$.

This in turn gives

$$|\lambda|(E) = \int_E |g|\, d\mu \qquad (E \in \mathcal{M}).$$ q.e.d.

Same as saying $|\mu|$ is σ-finite

The Hahn Decomposition Theorem: Suppose μ is a σ-finite extended signed measure. Then there exist sets A and B in $\mathcal{M}$ such that $A \cup B = X$, $A \cap B = \phi$ and such that

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = \mu(B \cap E) \qquad (E \in \mathcal{M}).$$

Proof:

Let $h = \frac{d|\mu|}{d\mu}$, with $|h| = 1$ everywhere. Since h is real (for $|\mu|$ and μ are), h only takes values $+1$ and -1. Let $A = \{h = 1\}$ and $B = \{h = -1\}$. It is clear that the the theorem is true with this A and B. q.e.d.

Corollary (Minimality of the Jordan decomposition): Suppose $\mu = \lambda_1 - \lambda_2$ where λ_1 and λ_2 are positive measures. Then $\mu^+ \le \lambda_1$ and $\mu^- \le \lambda_2$.

Proof:

Since $\mu \le \lambda_1$, we have

$$\mu^+(E) = \mu(E \cap A) \le \lambda_1(E \cap A) \le \lambda_1(E).$$ q.e.d.

Remark: The minimality of the Jordan decomposition shown in the last lecture has another consequence. Suppose, as in the Theorem, $d\lambda = g\,d\mu$ where $g \in L^1(\mu)$ (g being "locally $L^1(\mu)$" enough actually), and say g is real-valued, so that λ is a signed measure. Let σ and τ be the positive measures given by $d\sigma = g^+ d\mu$ and $d\tau = g^- d\mu$. Then $\lambda = \sigma - \tau$, whence $\sigma \geq \lambda^+$ and $\tau \geq \lambda^-$. Let $g_1 = \frac{d\lambda^+}{d\mu}$ and $g_2 = \frac{d\lambda^-}{d\mu}$. Then $\sigma \geq \lambda^+ \Rightarrow \int_E g^+ d\mu \geq \int_E g_1\,d\mu$ $\forall\, E \in \mathcal{M}$, whence $g^+ \geq g_1$ a.e. $[\mu]$. Similarly $g^- \geq g_2$ a.e. $[\mu]$. However $g^+ - g^- = g = g_1 - g_2$ whence, by the minimality of the decomposition $g = g^+ - g^-$ we have $g^+ = g_1$, $g^- = g_2$ a.e. $[\mu]$. After redefining on a μ-null set, we get

$$g^+ = \frac{d\lambda^+}{d\mu} \quad \text{and} \quad g^- = \frac{d\lambda^-}{d\mu}.$$

There are other ways of proving this, which are essentially similar. For example, since $|g|\,d\mu = d|\lambda|$, we have $g_1 + g_2 = |g| = g^+ + g^-$, a.e. $[\mu]$. This coupled with $g_1 - g_2 = g = g^+ - g^-$ a.e. $[\mu]$ forces $g_1 = g^+$ and $g_2 = g^-$ a.e. $[\mu]$.

Theorem (Lebesgue-Radon-Nikodym for σ-finite measures) Suppose μ is a σ-finite positive measure on $(X, \mathcal{M})$ and λ a σ-finite extended signed measure on $(X, \mathcal{M})$.

(a) There is a unique pair of complex measures λ_a and λ_s on $\mathcal{M}$ such that

$$\lambda = \lambda_a + \lambda_s, \quad \lambda_a \ll \mu, \quad \lambda_s \perp \mu.$$

(b) There is a measurable function $h : X \longrightarrow [-\infty, \infty]$ such that one of $\int_X h^+ d\mu$ or $\int_X h^- d\mu$ is finite and

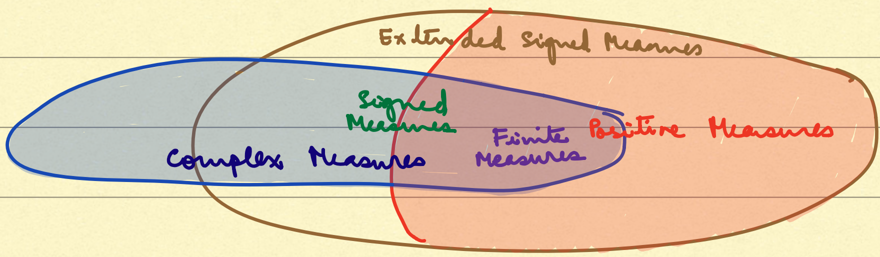
$$\lambda(E) = \int_E h \, d\mu .$$

Moreover $d\lambda^+ = h^+ d\mu$, $d\lambda^- = h^- d\mu$ and $d|\lambda| = |h| d\mu$.

Proof:

The proof is clear from the remark above. The σ-finiteness of λ (i.e., of $|\lambda|$) is used to break up X as $X = \bigcup_{n=1}^{\infty} E_n$, $|\lambda|(E_n) < \infty$, $n \in \mathbb{N}$, with E_n's being pairwise disjoint. The usual Lebesgue-Radon-Nikodym applies to $(\lambda|_{E_n}, \mu|_{E_n})$ giving the result. q.e.d.

Remark: Here is the relationship between complex, signed, extended signed, finite, and positive measures.



$$\boxed{\begin{array}{ccccc} CM & \supset & SM & \supset & FM \\ & & \cap & & \cap \\ & & ESM & \supset & PM \end{array}}$$

Signed measures = Real measures

$$\boxed{\begin{array}{l} ESM \cap CM = SM \\ SM \cap PM = CM \cap PM = FM \end{array}}$$

Integration with respect to a complex measure:

Let μ be a complex measure on $(X, \mathcal{M})$. A m'ble function $f : X \longrightarrow \mathbb{C}$ is said to be in $L^1(\mu)$ if it is in $L^1(|\mu|)$ and in this case, with $h = d\mu/d|\mu|$, we set

$$\int_X f \, d\mu = \int_X fh \, d|\mu|.$$

Since $|fh| = |f|$ and $f \in L^1(|\mu|)$ the R.S. is a well-defined complex number.

With this definition the relation

$$\mu(E) = \int_X \chi_E \, d\mu \qquad (E \in \mathcal{M})$$

holds. Moreover, using this definition, it is easy to see that

$$\int_X f \chi_E \, d\mu = \int_E (f|_E) \, d(\mu|_E) \qquad (E \in \mathcal{M}).$$

The right side is written in a less cumbersome way as $\int_E f \, d\mu$.