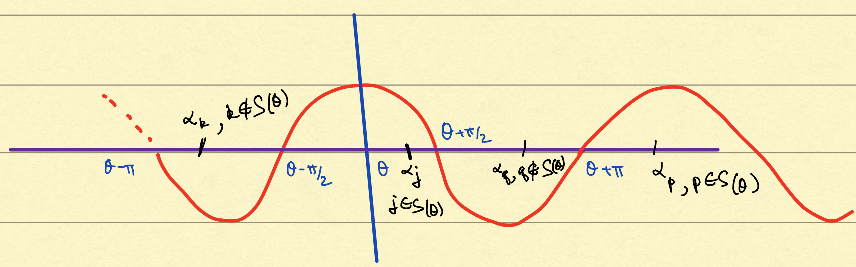
Lemma: Let $z_1, \dots, z_N \in \mathbb{C}$. There exists a subset S of $\{1, 2, \dots, N\}$ for which

$$\Big|\sum_{k\in S} z_k\Big| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

Remark: The factor $\frac{1}{\pi}$ is what allows us to "reverse" the triangle inequality. It is interesting that it is a universal constant, independent of N or of $z_1, \dots, z_N$.

Proof: Let α_k be an argument of z_k, so that $z_k = |z_k| e^{i\alpha_k}$.
For $\theta \in [-\pi, \pi]$, define

$$S(\theta) = \{ k \mid 1 \leq k \leq N,\ \cos(\alpha_k - \theta) > 0 \}$$



Then

$$\Big|\sum_{k\in S(\theta)} z_k\Big| = \Big|\sum_{k\in S(\theta)} e^{-i\theta} z_k\Big| \geq \mathrm{Re} \sum_{k\in S(\theta)} e^{-i\theta} z_k$$

$$= \sum_{k=1}^{n} |z_k| \cos^+(\alpha_k - \theta)$$

where $\cos^+ = \chi_{\{\cos > 0\}} \cos$

Let $\theta_0 \in [-\pi, \pi]$ be a point where $\sum_{k=1}^{n} |z_k| \cos^+(\alpha_k - \theta)$ achieves its maximum as a function of θ. Let $S = S(\theta)$.

Then

$$\left|\sum_{k\in S} z_k\right| \geq \sum_{k=1}^{n} |z_k| \cos^+(\alpha_k - \theta) \qquad \text{for every } \theta \in [-\pi, \pi].$$

Hence

$$\int_{-\pi}^{\pi} \left|\sum_{k\in S} z_k\right| d\theta \geq \sum_{k=1}^{N} |z_k| \int_{-\pi}^{\pi} \cos^+(\alpha_k - \theta)\, d\theta \qquad \text{---} (*)$$

Now $\int_{-\pi}^{\pi} \cos^+(\alpha - \theta)\, d\theta = 2$ whatever be θ since it is simply the area under the positive parts of the curve $y = \cos\theta$ in the θ-y plane over an interval of length 2π.

Thus $(*)$ gives

$$2\pi \left|\sum_{k\in S} z_k\right| \geq 2 \sum_{k=1}^{N} |z_k|$$

q.e.d.

Theorem: Let μ be a complex measure on $(X, \mathcal{M})$. Then the total variation $|\mu|$ is a finite measure.

Proof:

Suppose $E \in \mathcal{M}$ is such that $|\mu|(E) = \infty$. Let

$$t = \pi\,(1 + |\mu(E)|)$$

(so that $\frac{t}{\pi} \geq 1$). Since $|\mu|(E) = \infty$, there exists a m'ble partition $\{A_n\}$ of E s.t. $\sum_n |\mu(A_n)| > t$. In fact $\{A_n\}$ can be taken to be a finite partition of E, say $\{A_1, A_2, \ldots, A_N\}$. By the Lemma above, we have $S \subseteq \{1, \ldots, N\}$ such that

$$\left|\sum_{k\in S} \mu(A_k)\right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |\mu(A_k)| > \frac{t}{\pi} \geq 1.$$

Let $A = \bigcup_{k\in S} A_k$. The above shows that

$$\mu(A) > \frac{t}{\pi} \geq 1$$

Let $B = E - A$. Then

$$\mu(B) = \mu(E) - \mu(A) \geq |\mu(A)| - |\mu(E)|$$
$$> \frac{t}{\pi} - |\mu(E)|$$
$$= 1 \quad \text{(by definition of } t\text{)}.$$

Thus we can partition E into two m'ble sets A and B, with $\mu(A) > 1$, $\mu(B) > 1$, and either $\mu(A) = \infty$ or $\mu(B) = \infty$.

Thus X can be partitioned into A_1 and B_1, with $\mu(A_1) > 1$ and $\mu(B_1) = \infty$. Next B_1 can be partitioned into A_2 and B_2, $\mu(A_2) > 1$ and $\mu(B_2) = \infty$. Continue this. By induction we have disjoint m'ble sets $A_1, A_2, \ldots, A_n, \ldots$ with $\mu(A_i) > 1$ $\forall i \in \mathbb{N}$. This means $\sum_{n=1}^{\infty} \mu(A_n)$ is not convergent since $\lim_{n \to \infty} \mu(A_n) \neq 0$, which is a contradiction.

q.e.d.

Corollary: Let σ be an extended signed measure on (X, μ) and define $|\sigma| : \mathcal{M} \longrightarrow [0, \infty]$ in the same way that the total variation of a complex measure was defined. Then $|\sigma|$ is a positive measure.

Remark: We can no longer ensure that $|\sigma|$ is a finite measure.

Proof: This is not a direct corollary since extended signed measures need not be complex measures. The difficulty is when σ is not a signed measure. WLOG suppose σ takes value ∞ on some m'ble set. Then $\sigma(A) \neq -\infty$ for any $A \in \mathcal{M}$. This means that if $\sigma(B) = \infty$ for some $B \in \mathcal{M}$, then $\sigma(E) = \sigma(B) + \sigma(E - B) = \infty$ for every $E \in \mathcal{M}$ s.t. $E \supset B$, for $\sigma(E - B) \neq -\infty$.

In particular if $E \in \mathcal{M}$ is such that $\sigma(E) < \infty$, then $\sigma(A) < \infty$ for every m'ble subset A of E. It follows that σ restricted to E is a signed measure (i.e. a real valued complex measure) and from the theorem, if $\{A_n\}_{n=1}^{\infty}$ is a m'ble partition of E, then $|\sigma|(E) = \sum_{n=1}^{\infty} |\sigma|(A_n)$, and $|\sigma|(E) < \infty$.

Now suppose E is a m'ble set and $\{A_n\}_{n=1}^{\infty}$ is a m'ble partition of E such that $\sum_n |\sigma|(A_n) < \infty$. Then for every $N \in \mathbb{N}$

$$\sum_{n=1}^{N} \sigma(A_n) \leq \sum_{n=1}^{N} |\sigma(A_n)| \leq \sum_{n=1}^{N} |\sigma|(A_n) \leq \sum_{n=1}^{\infty} |\sigma|(A_n) < \infty.$$

It follows (by letting $N \to \infty$) that $\sigma(E) \leq \sum_n |\sigma|(A_n) < \infty$. Thus $\sigma(E) < \infty$ if and only if $\sum_n |\sigma|(A_n) < \infty$ for some (and hence every) m'ble partition $\{A_n\}_{n=1}^{\infty}$ of E.

Now suppose $\sigma(E) = \infty$. From above, $\sum_n |\sigma|(A_n) = \infty$ for every m'ble partition $\{A_n\}_{n=1}^{\infty}$ of E. In particular, setting $A_1 = E$, $A_n = \emptyset$, $n > 1$, $|\sigma|(E) = \infty$. Thus in this case too we have countable additivity. q.e.d.

Definition: Let σ be ~~an extended~~ a signed measure. Define

$$\sigma^+ = \frac{|\sigma| + \sigma}{2} \quad \text{and} \quad \sigma^- = \frac{|\sigma| - \sigma}{2}.$$

← Does not make sense for extended signed measures which have sets of infinite σ-value

The measures σ^+ and σ^- are positive measures ~~(at least one of them is finite)~~.
σ^+ is called the positive variation of σ and σ^- the negative variation. The decomposition

$$\boxed{\sigma = \sigma^+ - \sigma^-}$$

If σ is an extended signed measure such that $|\sigma|$ is σ-finite, then by breaking up X into a countable # sets on which $|\sigma|$ is finite, we can still define σ^+ & σ^-, and have the Jordan decomp.

is called the Jordan decomposition of σ. Note that $|\sigma| = \sigma^+ + \sigma^-$.
The Jordan decomposition (at least for σ-finite measures) is minimal in a sense we will describe later (hashtag: #HahnDecomposition).