October 2, 2018

Lecture 13

Complex Measures

Let $(X, \mathcal{M})$ be a measurable space. A map

$$\mu : \mathcal{M} \longrightarrow \mathbb{C}$$

is called a **complex measure** on $\mathcal{M}$ (or on X, or on $(X,\mathcal{M})$) if $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ for any countable sequence $E_1, E_2, \ldots, E_n, \ldots$ of pair-wise disjoint measurable sets.

Note two things:

1. Since the L.S. of the equality
$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n)$$
is independent of the arrangement of the sets $E_1, \ldots, E_n, \ldots$, therefore the R.S. is invariant under re-arrangement of terms. This means the sum $\sum_n \mu(E_n)$ is absolutely convergent.
2. If we set $E_n = \phi$, $n \in \mathbb{N}$, then we get $\mu(\phi) = \sum_{n=1}^{\infty} \mu(\phi)$ forcing $\mu(\phi) = 0$. From here, as before, finite additivity of μ follows.

The total variation of a complex measure:

There is a very important notion associated with a complex measure μ on $(X,\mathcal{M})$. Suppose $\{A_n\}$ is a measurable partition of $E \in \mathcal{M}$. Then $\exists\, \alpha_n \in \mathbb{C}$, $|\alpha_n| = 1$ such that $|\mu(A_n)| = \alpha_n \mu(A_n)$. This gives $\sum_n |\mu(A_n)| = \sum_n \alpha_n \mu(A_n)$. Intuitively taking some sort of limit, we should have that supremum of such sums over all partitions is $\int_E \alpha \, d\mu$, where α is a function with $|\alpha|=1$. The problem of course is that we don't have a theory of integrals with respect to complex measures (yet). But, going with the intuition, this suggests that $E \longmapsto \sup_{\{A_n\}} |\mu(A_n)|$, as $\{A_n\}$ varies over m'ble partitions of E gives us a measure since "integrals give measures". As is often the case in mathematics we reverse the process in formal proofs — first constructing a positive measure $|\mu|$ associated with μ, and then defining integrals w.r.t μ.

For $E \in \mathcal{M}$, let $\mathcal{P}(E)$ denote the set of countable or finite measurable partitions of E.

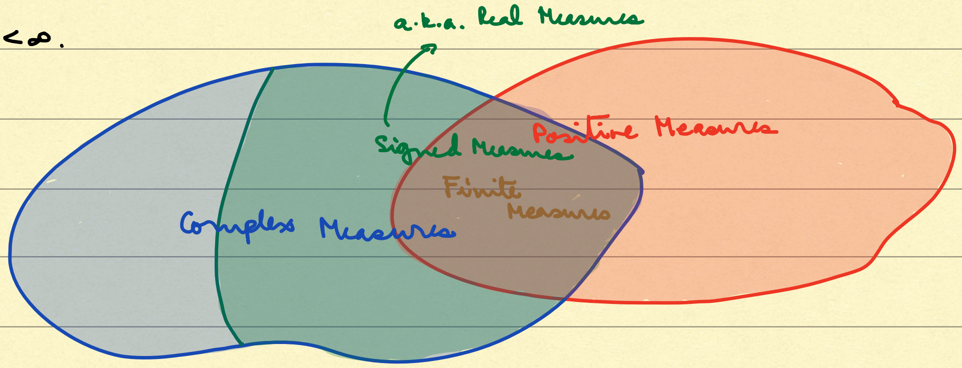
Define

$$|\mu|(E) = \sup_{\{A_n\} \in \mathcal{P}(E)} \sum_{n=1}^{\infty} |\mu(A_n)| .$$

Clearly $|\mu| : \mathcal{M} \longrightarrow [0,\infty]$.

We will show $|\mu|$ is a measure which is finite. $|\mu|$ is called the variation or the total variation of μ.

Terminology: Complex measures which take values in $\mathbb{R}$ are called real measures or more commonly signed measures. It is Tempting to call signed measure which take non-negative values as positive measures, but that term has already been reserved for measures taking values in $[0,\infty]$. Signed measures which take non-negative values are finite measures for such a measure μ must be a positive measure with $\mu(X) < \infty$.



$$CM \supset SM \supset FM \subset PM.$$

$$PM \cap SM = PM \cap CM = FM$$

Over and above that there is a notion of an extended signed measure. This is a countably additive function $\mu : \mathcal{M} \to [-\infty, \infty]$, $\mu(\phi) = 0$, such that μ cannot take value $-\infty$ if any set takes value ∞ (and vice-versa). For simplicity we will assume that if we have an extended signed measure then $\mu(E) \neq -\infty$ for any E, i.e., μ takes values in $(-\infty, \infty]$. With this convention, positive measures are a subset of extended signed measures.

Theorem: Let μ be a complex measure on $(X, \mathcal{M})$. The the total variation $|\mu|$ is a positive measure.

Proof:

Since $\mu(\phi) = 0$, clearly $|\mu|(\phi) = 0$. Hence we only need to check countable additivity for $|\mu|$. To that end let

$E \in \mathcal{M}$ and let $\{E_m\}_{m=1}^{\infty}$ be a m'ble partition of E. Pick real numbers t_m s.t. $t_m < |\mu|(E_m)$. Then each E_m has a partition $\{A_{m,n}\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |\mu(A_{m,n})| > t_m, \quad m \in \mathbb{N}.$$

Now $\{A_{m,n} \mid m,n \in \mathbb{N}\}$ is a partition of E. Hence

$$\sum_{m=1}^{\infty} t_m \le \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\mu(A_{m,n})| \le |\mu|(E).$$

It follows that

$$\sum_{m=1}^{\infty} |\mu|(E_m) \le |\mu|(E).$$

since $t_m < |\mu|(E_m)$ was arbitrary.

On the other hand, suppose $\{A_n\}_{n=1}^{\infty}$ is a m'ble partition of E. Then $\{E_m \cap A_n\}_{n=1}^{\infty}$ partitions of E_m whence

$$(*) \qquad \sum_{n=1}^{\infty} |\mu(E_m \cap A_n)| \le |\mu|(E_m), \quad m \in \mathbb{N}.$$

Thus

$$\sum_{n} |\mu(A_n)| = \sum_{n} \left| \sum_{m} \mu(E_m \cap A_n) \right|$$

$$\le \sum_{n} \sum_{m} |\mu(E_m \cap A_n)|$$

$$= \sum_{m} \sum_{n} |\mu(E_m \cap A_n)|$$

$$\le \sum_{m} |\mu|(E_m) \qquad \text{(via } (*)\text{)}.$$

Since $\{A_n\}$ was an arbitrary m'ble partition of E, we get

$$|\mu|(E) \le \sum_{m} |\mu|(E_m).$$

q.e.d.