October 2, 2018

Lecture 13

Complex Measures

Let $(X, \mathcal{M})$ be a measurable space. A map

$$\mu : \mathcal{M} \longrightarrow \mathbb{C}$$

is called a **complex measure** on $\mathcal{M}$ (or on X, or on $(X, \mathcal{M})$) if $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ for any countable sequence $E_1, E_2, \ldots, E_n, \ldots$ of pair-wise disjoint measurable sets.

Note two things:

1. Since the L.S. of the equality
$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n)$$
is independent of the arrangement of the sets $E_1, \ldots, E_n, \ldots$, therefore the R.S. is invariant under re-arrangement of terms. This means the sum $\sum_n \mu(E_n)$ is absolutely convergent.

2. If we set $E_n = \phi$, $n \in \mathbb{N}$, then we get $\mu(\phi) = \sum_{n=1}^{\infty} \mu(\phi)$ forcing $\mu(\phi) = 0$. From here, as before, finite additivity of μ follows.

The total variation of a complex measure:

There is a very important notion associated with a complex measure μ on $(X, \mathcal{M})$. Suppose $\{A_n\}$ is a measurable partition of $E \in \mathcal{M}$. Then $\exists\, \alpha_n \in \mathbb{C}$, $|\alpha_n| = 1$ such that $|\mu(A_n)| = \alpha_n \mu(A_n)$. This gives $\sum_n |\mu(A_n)| = \sum_n \alpha_n \mu(A_n)$. Intuitively taking some sort of limit, we should have that supremum of such sums over all partitions is $\int_E \alpha \, d\mu$, where α is a function with $|\alpha|=1$. The problem of course is that we don't have a theory of integrals with respect to complex measures (yet). But, going with the intuition, this suggests that $E \longmapsto \sup_{\{A_n\}} |\mu(A_n)|$, as $\{A_n\}$ varies over m'ble partitions of E gives us a measure since "integrals give measures". As is often the case in mathematics we reverse the process in formal proofs — first constructing a positive measure $|\mu|$ associated with μ, and then defining integrals w.r.t μ.

For $E \in \mathcal{M}$, let $\mathcal{P}(E)$ denote the set of countable or finite measurable partitions of E.

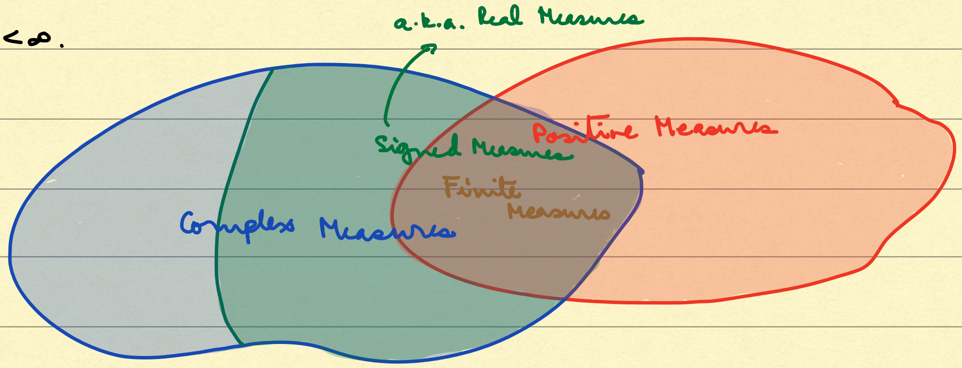
Define

$$|\mu|(E) = \sup_{\{A_n\} \in \mathcal{P}(E)} \sum_{n=1}^{\infty} |\mu(A_n)|.$$

Clearly $|\mu| : \mathcal{M} \longrightarrow [0, \infty]$.

We will show $|\mu|$ is a measure which is finite. $|\mu|$ is called the variation or the total variation of μ.

Terminology: Complex measures which take values in $\mathbb{R}$ are called real measures or more commonly signed measures. It is tempting to call signed measure which take non-negative values as positive measures, but that term has already been reserved for measures taking values in $[0, \infty]$. Signed measures which take non-negative values are finite measures for such a measure μ must be a positive measure with $\mu(X) < \infty$.



$$CM \supset SM \supset FM \subset PM.$$

$$PM \cap SM = PM \cap CM = FM$$

Over and above that there is a notion of an extended signed measure. This is a countably additive function $\mu : \mathcal{M} \to [-\infty, \infty]$, $\mu(\phi) = 0$, such that μ cannot take value $-\infty$ if any set takes value ∞ (and vice-versa). For simplicity we will assume that if we have an extended signed measure then $\mu(E) \neq -\infty$ for any E, i.e., μ takes values in $(-\infty, \infty]$. With this convention, positive measures are a subset of extended signed measures.

Theorem: Let μ be a complex measure on $(X, \mathcal{M})$. The the total variation $|\mu|$ is a positive measure.

Proof:

Since $\mu(\phi) = 0$, clearly $|\mu|(\phi) = 0$. Hence we only need to check countable additivity for $|\mu|$. To that end let

$E \in \mathcal{M}$ and let $\{E_m\}_{m=1}^{\infty}$ be a m'ble partition of E. Pick real numbers t_m s.t. $t_m < |\mu|(E_m)$. Then each E_m has a partition $\{A_{m,n}\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |\mu(A_{m,n})| > t_m, \quad m \in \mathbb{N}.$$

Now $\{A_{m,n} \mid m,n \in \mathbb{N}\}$ is a partition of E. Hence

$$\sum_{m=1}^{\infty} t_m \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\mu(A_{m,n})| \leq |\mu|(E).$$

It follows that

$$\sum_{m=1}^{\infty} |\mu|(E_m) \leq |\mu|(E).$$

since $t_m < |\mu|(E_m)$ was arbitrary.

On the other hand, suppose $\{A_n\}_{n=1}^{\infty}$ is a m'ble partition of E. Then $\{E_m \cap A_n\}_{n=1}^{\infty}$ partitions E_m whence

$$(*) \qquad \sum_{n=1}^{\infty} |\mu(E_m \cap A_n)| \leq |\mu|(E_m), \quad m \in \mathbb{N}.$$

Thus

$$\sum_{n} |\mu(A_n)| = \sum_{n} \left| \sum_{m} \mu(E_m \cap A_n) \right|$$

$$\leq \sum_{n} \sum_{m} |\mu(E_m \cap A_n)|$$

$$= \sum_{m} \sum_{n} |\mu(E_m \cap A_n)|$$

$$\leq \sum_{m} |\mu|(E_m) \qquad (\text{via } (*)).$$

Since $\{A_n\}$ was an arbitrary m'ble partition of E, we get

$$|\mu|(E) \leq \sum_{m} |\mu|(E_m).$$

q.e.d.

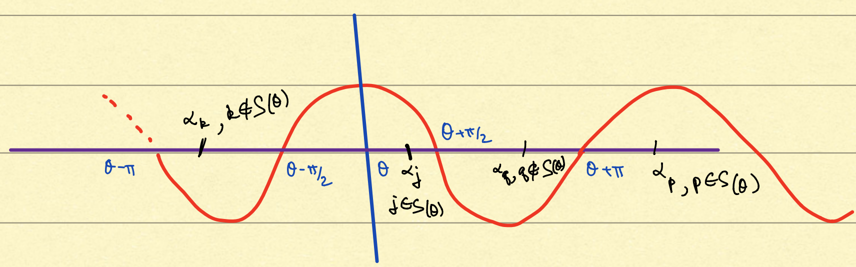
Lemma: Let $z_1, \dots, z_N \in \mathbb{C}$. There exists a subset S of $\{1, 2, \dots, N\}$ for which

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

Remark: The factor $\frac{1}{\pi}$ is what allows us to "reverse" the triangle inequality. It is interesting that it is a universal constant, independent of N or of $z_1, \dots, z_N$.

Proof: Let α_k be an argument of z_k, so that $z_k = |z_k| e^{i\alpha_k}$.
For $\theta \in [-\pi, \pi]$, define

$$S(\theta) = \{ k \mid 1 \leq k \leq N, \ \cos(\alpha_k - \theta) > 0 \}$$



Then

$$\left| \sum_{k \in S(\theta)} z_k \right| = \left| \sum_{k \in S(\theta)} e^{-i\theta} z_k \right| \geq \mathrm{Re} \sum_{k \in S(\theta)} e^{-i\theta} z_k$$

$$= \sum_{k=1}^{n} |z_k| \cos^+(\alpha_k - \theta)$$

where $\cos^+ = \chi_{\{\cos > 0\}} \cos$

Let $\theta_0 \in [-\pi, \pi]$ be a point where $\sum_{k=1}^{n} |z_k| \cos^+(\alpha_k - \theta)$ achieves its maximum as a function of θ. Let $S = S(\theta)$.

Then

$$\left|\sum_{k\in S} z_k\right| \geq \sum_{k=1}^{n} |z_k| \cos^+(\alpha_k - \theta) \quad \text{for every } \theta \in [-\pi, \pi].$$

Hence

$$\int_{-\pi}^{\pi} \left|\sum_{k\in S} z_k\right| d\theta \geq \sum_{k=1}^{N} |z_k| \int_{-\pi}^{\pi} \cos^+(\alpha_k - \theta)\, d\theta \qquad \text{—— } (*)$$

Now $\int_{-\pi}^{\pi} \cos^+(\alpha - \theta)\, d\theta = 2$ whatever be θ since it is simply the area under the positive parts of the curve $y = \cos\theta$ in the θ-y plane over an interval of length 2π.

Thus $(*)$ gives

$$2\pi \left|\sum_{k\in S} z_k\right| \geq 2 \sum_{k=1}^{N} |z_k|$$

q.e.d.

Theorem: Let μ be a complex measure on $(X, \mathcal{M})$. Then the total variation $|\mu|$ is a finite measure.

Proof:

Suppose $E \in \mathcal{M}$ is such that $|\mu|(E) = \infty$. Let

$$t = \pi\,(1 + |\mu(E)|)$$

(so that $\frac{t}{\pi} \geq 1$). Since $|\mu|(E) = \infty$, there exists a m'ble partition $\{A_n\}$ of E s.t. $\sum_n |\mu(A_n)| > t$. In fact $\{A_n\}$ can be taken to be a finite partition of E, say $\{A_1, A_2, \ldots, A_N\}$. By the Lemma above, we have $S \subseteq \{1, \ldots, N\}$ such that

$$\left|\sum_{k\in S} \mu(A_k)\right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |\mu(A_k)| > \frac{t}{\pi} \geq 1.$$

Let $A = \bigcup_{k\in S} A_k$. The above shows that

$$\mu(A) > \frac{t}{\pi} \geq 1$$

Let $B = E \sim A$. Then

$$\mu(B) = \mu(E) - \mu(A) \geq |\mu(A)| - |\mu(E)|$$
$$> \frac{t}{\pi} - |\mu(E)|$$
$$= 1 \quad \text{(by definition of } t\text{)}.$$

Thus we can partition E into two m'ble sets A and B, with $\mu(A) > 1$, $\mu(B) > 1$, and either $\mu(A) = \infty$ or $\mu(B) = \infty$.

Thus X can be partitioned into A_1 and B_1, with $\mu(A_1) > 1$ and $\mu(B_1) = \infty$. Next B_1 can be partitioned into A_2 and B_2, $\mu(A_2) > 1$ and $\mu(B_2) = \infty$. Continue this. By induction we have disjoint m'ble sets $A_1, A_2, \ldots, A_n, \ldots$ with $\mu(A_i) > 1$ $\forall\, i \in \mathbb{N}$. This means $\sum_{n=1}^{\infty} \mu(A_n)$ is not convergent since $\lim_{n\to\infty} \mu(A_n) \neq 0$, which is a contradiction.

q.e.d.

Corollary: Let σ be an extended signed measure on $(X, \mathcal{M})$ and define $|\sigma| : \mathcal{M} \longrightarrow [0, \infty]$ in the same way that the total variation of a complex measure was defined. Then $|\sigma|$ is a positive measure.

Remark: We can no longer ensure that $|\sigma|$ is a finite measure.

Proof: This is not a direct corollary since extended signed measures need not be complex measures. The difficulty is when σ is not a signed measure. WLOG suppose σ takes value ∞ on some m'ble set. Then $\sigma(A) \neq -\infty$ for any $A \in \mathcal{M}$. This means that if $\sigma(B) = \infty$ for some $B \in \mathcal{M}$, then $\sigma(E) = \sigma(B) + \sigma(E - B) = \infty$ for every $E \in \mathcal{M}$ s.t. $E \supset B$, for $\sigma(E - B) \neq -\infty$.

In particular if $E \in \mathcal{M}$ is such that $\sigma(E) < \infty$, then $\sigma(A) < \infty$ for every m'ble subset A of E. It follows that σ restricted to E is a signed measure (i.e. a real valued complex measure) and from the theorem, if $\{A_n\}_{n=1}^{\infty}$ is a m'ble partition of E, then $|\sigma|(E) = \sum_{n=1}^{\infty} |\sigma|(A_n)$, and $|\sigma|(E) < \infty$.

Now suppose E is a m'ble set and $\{A_n\}_{n=1}^{\infty}$ is a m'ble partition of E such that $\sum_n |\sigma|(A_n) < \infty$. Then for every $N \in \mathbb{N}$

$$\sum_{n=1}^{N} \sigma(A_n) \leq \sum_{n=1}^{N} |\sigma(A_n)| \leq \sum_{n=1}^{N} |\sigma|(A_n) \leq \sum_{n=1}^{\infty} |\sigma|(A_n) < \infty.$$

It follows (by letting $N \to \infty$) that $\sigma(E) \leq \sum_n |\sigma|(A_n) < \infty$. Thus $\sigma(E) < \infty$ if and only if $\sum_n |\sigma|(A_n) < \infty$ for some (and hence every) m'ble partition $\{A_n\}_{n=1}^{\infty}$ of E.

Now suppose $\sigma(E) = \infty$. From above, $\sum_n |\sigma|(A_n) = \infty$ for every m'ble partition $\{A_n\}_{n=1}^{\infty}$ of E. In particular, setting $A_1 = E$, $A_n = \emptyset$, $n > 1$, $|\sigma|(E) = \infty$. Thus in this case too we have countable additivity. q.e.d.

Definition: Let σ be ~~an extended~~ a signed measure. Define

$$\sigma^+ = \frac{|\sigma| + \sigma}{2} \quad \text{and} \quad \sigma^- = \frac{|\sigma| - \sigma}{2}.$$ ← Does not make sense for extended signed measures which have sets of infinite σ-value

The measures σ^+ and σ^- are positive measures ~~(at least one of them is finite)~~. σ^+ is called the positive variation of σ and σ^- the negative variation. The decomposition

$$\boxed{\sigma = \sigma^+ - \sigma^-}$$

If σ is an extended signed measure such that $|\sigma|$ is σ-finite, then by breaking up X into a countable # sets on which $|\sigma|$ is finite, we can still define σ^+ & σ^-, and have the Jordan decomp.

is called the Jordan decomposition of σ. Note that $|\sigma| = \sigma^+ + \sigma^-$. The Jordan decomposition (at least for σ-finite measures) is minimal in a sense we will describe later (hashtag: #HahnDecomposition).