Sep 20, 2018

Lecture 12

Throughout this lecture $(X, \mathcal{M}, \mu)$ is a measure space. When we say "measurable" we mean "measurable w.r.t $\mathcal{M}$", and "a.e." will mean "a.e. $[\mu]$".

Essential supremum: Let $g : X \longrightarrow [0,\infty]$ be a m'ble function. Let S be the set of $\alpha \in [0,\infty)$ such that

$$\mu\left(g^{-1}((\alpha,\infty])\right) = 0.$$

If $S = \phi$, set $\beta = \infty$. Otherwise set $\beta = \inf S$. If $\beta < \infty$, then $(\beta, \infty] = \bigcup_{n=1}^{\infty} (\beta + \alpha_n, \infty]$, where $\alpha_n \downarrow \beta$, $\alpha_n \in S$. Hence $\beta \in S$.

β is called the essential supremum of g.

For measurable f, $\|f\|_\infty$ denotes the essential supremum of $|f|$.

It is clear that $|f| \leq \lambda$ a.e. if and only if $\|f\|_\infty \leq \lambda$. In particular $\|f\|_\infty = 0$ if and only if $f = 0$ a.e.

Equivalence relation: We have already seen that $\|f\|_\infty = 0$ is equivalent to saying $f = 0$ a.e. Similarly, if $1 \leq p < \infty$,

$$\|f\|_p = 0 \iff \int_X |f|^p \, d\mu = 0$$

$$\iff |f|^p = 0 \text{ a.e.} \quad (\text{From exercise in Quiz 3})$$

$$\iff f = 0 \text{ a.e.}$$

In particular, $\|f-g\|_p = 0$ if and only $f = g$ a.e. (this includes the $p = \infty$ case). Now "equality a.e. $[\mu]$" is an equivalence relation on the set of measurable functions f with $\|f\|_p < \infty$. For $1 \le p \le \infty$, let $L^p(\mu)$ denote the set of equivalence classes of m'ble funtions f with $\|f\|_p < \infty$. As we discussed in earlier notes and lectures, we continue to write $f \in L^p(\mu)$ and think of f as a m'ble funtion with $\|f\|_p < \infty$, as well as an equivalence class. This abuse of notation is standard and causes very little problems, provided one knows that one is really dealing with equivalence classes.

Theorem: Let $1 \le p \le \infty$. Let q be the conjugate exponent, i.e., q satisfies $\frac{1}{p} + \frac{1}{q} = 1$ if $1 < p < \infty$; $q = \infty$ if $p = 1$; $q = 1$ if $p = \infty$. Then

(a) If $f \in L^p(\mu)$, $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and

$$\|fg\|_1 \le \|f\|_p \, \|g\|_q$$

(b) $\|f+g\|_p \le \|f\|_p + \|g\|_p \qquad f, g \in L^p(\mu)$

(c) $\|\alpha f\|_p = |\alpha| \, \|f\|_p \quad , \quad \alpha \in \mathbb{C}$.

Proof:

Part (c) is obvious.

For $1 < p < \infty$, (a) and (b) follow from Hölder's and Minkowski's eincqualities respectively. For $p = 1$, (a) follows from $|fg| \le |f| \cdot \|g\|_\infty$ and for $p = \infty$ from $|fg| \le \|f\|_\infty \cdot |g|$. As for (b), the cases $p = 1$ and $p = \infty$ both follow from $|f+g| \le |f| + |g|$. q.e.d.

Corollary: $(L^p(\mu), \| \|_p)$ is a normed linear space.

Proof: From (b) and (c) it is clear that $L^p(\mu)$ is closed under addition and scalar multiplication (whence under the operation of taking negatives). Therefore it is a vector space over $\mathbb{C}$. Moreover $\|f\|_p = 0$ if and only $f=0$ in $L^p(\mu)$ (i.e. $f=0$ a.e.). From this and (b) and (c) of the Theorem, the Corollary follows. q.e.d.

Theorem: $L^p(\mu)$ is a Banach space for $1 \le p \le \infty$.

Proof:

Let us first consider the case $1 \le p < \infty$. Suppose $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$. We can find a subsequence $\{f_{n_i}\}$ such that

$$\| f_{n_{i+1}} - f_{n_i} \|_p \le \frac{1}{2^i}$$

Let

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| , \quad g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|.$$

Now $g_k \in L^p(\mu)$ for each k and g_k increases to g. This means $g_k^p \uparrow g^p$. By MCT $\|g_k\|_p^p \longrightarrow \|g\|_p^p$. Clearly $\|g_k\|_p \le 1$. It follows that $\|g\|_p \le 1$. From this one concludes that the series defining g converges outside a set $B \in \mathfrak{M}$, with $\mu(B) = 0$. Then for $x \notin B$ the series $\sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x))$ converges absolutely. Note that $\sum_{i=1}^{k} (f_{n_{i+1}} - f_{n_i}) = f_{n_{k+1}} - f_{n_1}$, $k \in \mathbb{N}$.

$$f(x) = \begin{cases} 0 & \text{if } x \in B \\ f_{n_1}(x) + \sum_{i=1}^{\infty} \left(f_{n_{i+1}}(x) - f_{n_i}(x) \right), & x \notin B. \end{cases}$$

It follows that $\lim_{i \to \infty} f_{n_i} = f$ a.e. (everywhere on $X - B$).
We claim that $f \in L^p(\mu)$ and that $f_n \longrightarrow f$ in $L^p(\mu)$ as $n \longrightarrow \infty$. Now given $\varepsilon > 0$ there exists N such that $\|f_n - f_m\| < \varepsilon$ for $n, m \geq N$.

Then for $n \geq N$:

$$\int_X |f_n - f|^p \, d\mu = \int_X \lim_{i \to \infty} |f_n - f_{n_i}|^p \, d\mu$$

$$\leq \overline{\lim_{i \to \infty}} \int_X |f_n - f_{n_i}|^p \, d\mu \quad \text{(Fatou's Lemma)}$$

$$= \overline{\lim_{i \to \infty}} \|f_n - f_{n_i}\|_p^p$$

$$\leq \varepsilon^p.$$

This shows that $f - f_n \in L^p(\mu)$ for $n \geq N$, whence $f = (f - f_N) + f_N \in L^p(\mu)$, and since $\|f_n - f\| \leq \varepsilon$ for $n \geq N$, $\{f_n\}$ converges to f in $L^p(\mu)$.

The case $p = \infty$ remains. To that end, suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty(\mu)$. Let $A_k = \{ |f_k| > \|f_k\|_\infty \}$ and $B_{m,n} = \{ |f_n - f_m| > \|f_n - f_m\|_\infty \}$. Then $\mu(A_k) = \mu(B_{m,n}) = 0$ for $k, m, n \in \mathbb{N}$. Let $E = \bigcup_{k \geq 1} A_k \cup \bigcup_{m,n \geq 1} B_{m,n}$. Then $\mu(E) = 0$. On $X - E$, $\{f_n\}$ converges uniformly to a bounded function f. Set $f(x) = 0$ for $x \in E$. Then $f \in L^\infty(\mu)$ and $\|f_n - f\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.

q.e.d.

The proof contains an important result.

Theorem: If $1 \le p \le \infty$ and $\{f_n\}$ is Cauchy in $L^p(\mu)$, with limit f, then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to f.

Simple functions and $L^p(\mu)$

Theorem: Let S be the set of all complex, measurable, simple functions on X such that

$$\mu(\{s \neq 0\}) < \infty \qquad (*)$$

If $1 \le p < \infty$, then S is dense in $L^p(\mu)$.

Proof:

It is clear that $S \subset L^p(\mu)$.

Suppose $f \geq 0$, $f \in L^p(\mu)$. Pick $\{s_n\}$, s_n simple, $0 \le s_n \le f$, $s_n \uparrow f$ as we did before. Since $f \in L^p(\mu)$, therefore $s_n \in L^p(\mu)$, $n \in \mathbb{N}$. From here it is not hard to show that $s_n \in S$ for each $n \in \mathbb{N}$. Now $|f - s_n|^p \le f^p$, and hence by DCT, $\|f - s_n\|_p \longrightarrow 0$ as $n \longrightarrow \infty$. Thus f is in the closure of S in $L^p(\mu)$. The general (with f complex) follows easily from this.

q.e.d.