It follows that, since $R^\circ \subset R \subset \overline{R}$, that

$$\boxed{m(R) = \delta_1 \cdots \delta_n = \text{vol}\, R.}$$

In fact for any S s.t. $R^\circ \subset S \subset \overline{R}$, $m(S) = \delta_1 \cdots \delta_n$.

In particular the faces of $\overline{R}$ have Lebesgue measure 0.

This in turn means:

Proposition: For $i = 1, \dots, n$ let $S_i = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid a_i = 0\}$.
Then $m(S_i) = 0$.

Proof: This is an immediate consequence of the countable additivity of m, and the fact that the faces of $\overline{R}$ have m measure zero. q.e.d.

Translation invariant measures:

As we have seen, m is translation invariant on $\mathcal{L}$.

Suppose μ is translation invariant on $\mathcal{B}(X)$.

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\delta_1, \delta_2, \dots, \delta_n$ be positive real numbers. Let $R = R(a; \delta_1, \dots, \delta_n)$. Fix a positive real number k k, and write $R' = R(a; k\delta_1, \delta_2, \dots, \delta_n)$.

If $k \in \mathbb{N}$, then R' can be written as the disjoint union

$$R' = R \cup (R + e_1) \cup (R + 2e_1) \cup \dots \cup (R + (k-1)e_1)$$

where $e_1 = (1, 0, \dots, 0)$. Since μ is translation invariant, we have

$$\mu(R') = k\,\mu(R)$$

If $k = \frac{1}{\ell}$, then R can be written as the disjoint union of $R' + \frac{j}{\ell} e_1$, $j = 0, \dots, \ell - 1$, whence $\mu(R) = \ell\,\mu(R')$,

and hence once again we have
$$\mu(R') = k\mu(R).$$
Combining the two we see
$$\mu(R') = k(\mu) \quad \text{for } k \in \mathbb{Q} \cap [0,\infty).$$
If $k_m \uparrow k$, $k_m > 0$ $\forall m$, $k > 0$, then clearly the rectangles $R_m = R(a; k_m\delta_1, \delta_2, \dots, \delta_n)$ increase to R'. Since k can be written as the increasing limit of rationals this means
$$\mu(R') = \lim_{m\to\infty} \mu(R_m) = \lim_{m\to\infty} k_m \mu(R) = k\mu(R).$$

A little thought shows that this argument can be repeated with δ_1 replaced by δ_j for any j. Hence we have

(*) — $$\boxed{\mu(a; k_1\delta_1, \dots, k_n\delta_n) = k_1 \cdots k_n \mu(a; \delta_1, \dots, \delta_n) \quad \left\}\begin{array}{l} a \in \mathbb{R}^n \\ k_i > 0 \ \ i = 1, \dots, n \\ \delta_i > 0 \ \ i = 1, \dots, n \end{array}\right.}$$

In view of (*) we have the following

Theorem: Let $Q_0 = Q(0;1)$ and $c = \mu(Q_0)$ where μ is a translation invariant positive measure on $\mathcal{B}(X)$
$$\mu = cm\big|_{\mathcal{B}(X)}$$
where m is the Lebesgue measure on $\mathbb{R}^n$.

Proof: By (*) we see that for a δ-box Q, we have

$\mu(Q) = \delta^n \mu(Q_0) = c\delta^n = c\, m(Q_0)$. The result follows from earlier results (see Proposition towards the end of the previous lecture).

Proposition: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, $\mathcal{L}$ the Lebesgue σ-algebra on $\mathbb{R}^n$ and m the Lebesgue measure on $\mathbb{R}^n$. Let $\mu: \mathcal{L} \longrightarrow [0,\infty]$ be the measure

$$\mu(E) := m(T(E)) \quad , \quad E \in \mathcal{L}.$$

Then there exists a non-negative scalar $\Delta(T)$ such that

$$\mu(E) = \Delta(T)\, m(E).$$

Proof:

Clearly μ is translation invariant. From the previous result, if $\Delta T = \mu(Q_0)$ then $\mu(E) = \Delta(T)\, m(E)$ for $E \in \mathcal{B}(X)$. For a general $E \in \mathcal{L}$, since $\mathbb{R}^n$ is σ-compact, there exists an F_σ set A and a G_δ set B such that $A \subset E \subset B$ and $m(B-A) = 0$. Now A, B are Borel sets and hence $\mu(A) = \Delta(T)\, m(A)$ and $\mu(B) = \Delta(T)\, m(B)$. Thus

$$\Delta(T) m(E) = \Delta(T) m(A) = \mu(A) \le \mu(E) \le \mu(B) = \Delta(T)\, m(B) = \Delta(T)\, m(E).$$

It follows that $\mu(E) = \Delta(T)\, m(E)$.

Remark: It is easy to see that $\Delta(T) = |\det T|$ by seeing this is true for elementary linear transformations and then noting that every linear transformation is the product of elementary linear transformations. In greater detail, it is clear that

$\Delta(T_1 T_2) = \Delta(T_1)\,\Delta(T_2)$ for any two linear transformations T_1 and T_2 and $|\det(T_1 T_2)| = |\det T_1|\,|\det T_2|$. Now T is the product of the following types of linear transformation S

(a) $\{Se_1, \dots, Se_n\}$ is a permutation of $\{e_1, \dots, e_n\}$

(b) $Se_1 = \alpha e_1$, $Te_i = e_i$, $i = 2, \dots, n$

(c) $Se_1 = e_1 + e_2$, $Se_i = e_i$, $e = 2, \dots, n$.

For S as in (a), $S(Q_0) = Q_0$, whence $\Delta(S) = 1$. On the other hand $|\det S|$ is also equal to 1.

For (b) $S(Q_0) = R(0; \alpha, 1, \dots, 1)$ if $\alpha > 0$. Hence $m(S(Q_0)) = \alpha = |\det S|$. If $\alpha = 0$, $S(Q_0)$ lies in $\{(a_1, \dots, a_n) \mid a_1 = 0\}$, whence from an earlier Proposition, $m(S(Q_0))$, i.e., $\Delta(S) = 0$. On the other hand if $\alpha = 0$, clearly $\det S = 0$. If $\alpha < 0$, then $S(Q_0)$ is sandwiched between $(\alpha, 0) \times \overbrace{[0,1) \times \dots \times [0,1)}^{n-1 \text{ factors}}$ and $[\alpha, 0] \times \overbrace{[0,1] \times \dots \times [0,1]}^{n-1 \text{ factors}}$ and hence $m(S(Q_0)) = -\alpha = |\alpha| = |\det S|$. Thus $\Delta(S) = |\det S|$ in this case too.

(c) In this case $S(Q_0)$ is a <u>disjoint union</u>

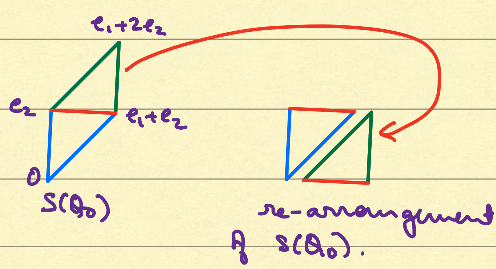
$$S(Q_0) = A \sqcup (B + e_2)$$

where

$$A = \{(x_1, \dots, x_n) \mid 0 \le x_1 \le x_2,\ 0 \le x_i < 1,\ i = 2, \dots, n\}$$

and

$$B = \left\{(x_1, \dots, x_n) \;\middle|\; \begin{array}{l} 0 \le x_2 < x_1,\ 0 < x_1 < 1 \\ 0 \le x_i < 1,\ i = 3, \dots, n \end{array}\right\}$$



Thus $m(S(Q_0)) = m(A) + m(B + e_1)$

$= m(A) + m(B)$ (by translation invariance of m)

$= m(A \cup B) = m(Q_0) = 1. \Rightarrow \Delta(T) = 1 = |\det S|.$

This proves the assertion. q.e.d.