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Lecture 10

The Lebesgue measure on $\mathbb{R}^n$

As mentioned earlier, let

$$\Lambda : C_c(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

be given by

$$\Lambda f = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n \quad , \; f \in C_c(\mathbb{R}^n)$$

where the right side is the Riemann integral of f over a closed rectangle containing Supp f. The answer is clearly independent of the rectangle of integration chosen.

Let $(\mathcal{L}, m)$ be the corresponding complete σ-algebra and measure. $\mathcal{L}$ is called the Lebesgue σ-algebra on $\mathbb{R}^n$ and $m : \mathcal{L} \longrightarrow [0, \infty]$ the Lebesgue measure. If we wish to emphasize the role of n, we write $\mathcal{L}_n$ and μ_n for $\mathcal{L}$ and μ.

Since every open set in $\mathbb{R}^n$ is σ-compact, m is clearly regular.

For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $\tau_a : C_c(\mathbb{R}^n) \longrightarrow C_c(\mathbb{R}^n)$ be

$$(\tau_a f)(x) = f(x - a) \qquad x \in \mathbb{R}^n.$$

Then

$$(1) \qquad \int_{\mathbb{R}^n} (\tau_a f)(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n.$$

In fact, if $I_1, \ldots, I_n$ are closed bounded intervals in $\mathbb{R}$ and $R = I_1 \times \cdots \times I_n$, then, as is well-known

$$\int_R f(t_1,\dots,t_n)\,dt_1\dots dt_n = \int_{R+a} (\tau_a f)(t_1,\dots,t_n)\,dt_1\dots dt_n .$$

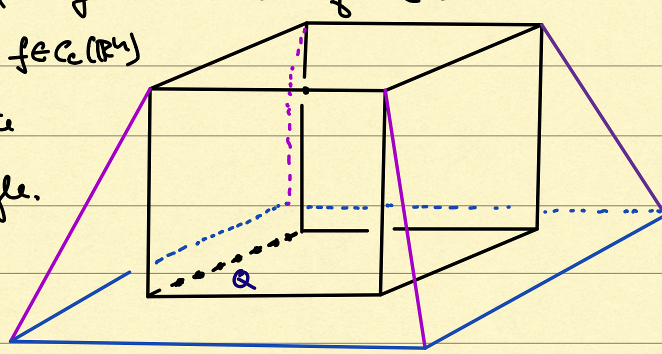
In somewhat greater detail, for $f \in C_c(\mathbb{R}^n)$, K a compact subset of $\mathbb{R}^n$, and V an open subset of $\mathbb{R}^n$ we have

- $\operatorname{Supp} \tau_a f = (\operatorname{Supp} f) + a$
- $\{g \in C_c(\mathbb{R}^n) \mid g \prec V+a\} = \{g = \tau_a h \mid h \in C_c(\mathbb{R}^n),\ h \prec V\}$
- $\{g \in C_c(\mathbb{R}^n) \mid K+a \prec g\} = \{g = \tau_a h \mid h \in C_c(\mathbb{R}^n),\ K \prec h\}$

From the above and (1) and the construction of $(\mathcal{L}, m)$ from Λ, it is easy to see that

$$\boxed{m(E+a) = m(E), \quad E \in \mathcal{L}.}$$

To understand the Lebesgue measure we need to work out measures of δ-boxes and various rectangles. Recall that in the last class we showed that $\mathcal{B}(X)$ has a subset Ω consisting of 2^{-n}-boxes, $n \in \mathbb{N}$, such that if μ, ν are Borel measures s.t., $\mu(Q) = \nu(Q) < \infty$ $\forall\, Q \in \Omega$, then $\mu = \nu$. Working out $m(Q)$ needs us to approximate χ_Q by elements of $C_c(\mathbb{R}^n)$. The picture displayed is of graph of $f \in C_c(\mathbb{R}^n)$ with $\bar{Q} \prec f \prec V$, where V is the open set outside the blue rectangle. As the blue rectangle approaches $\bar{Q}$, $f \longrightarrow \chi_{\bar{Q}}$. ($\bar{Q}$ = closure of Q)



Notations and terminology

Fix $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Recall that a δ-box with corner a, for a positive real number a, is

$$Q(a,\delta) = [a_1, a_1+\delta) \times \cdots \times [a_n, a_n+\delta).$$

We also define the closed δ-box with corner a, $\bar{Q}(a,\delta)$ and the open δ-box with corner a, $Q^\circ(a,\delta)$ to be

$$\bar{Q}(a,\delta) = [a_1, a_1+\delta] \times \cdots \times [a_n, a_n+\delta]$$

and $$Q^\circ(a,\delta) = (a_1, a_1+\delta) \times \cdots \times (a_n, a_n+\delta).$$

There are of course many sets between $Q^\circ(a,\delta)$ and $\bar{Q}(a,\delta)$ other than $Q(a,\delta)$ but for the moment these three boxes are all we need.

$$Q^\circ(a,\delta) \subset Q(a,\delta) \subset \bar{Q}(a,\delta).$$

If a and δ are understood from the context, we write $Q^\circ, Q, \bar{Q}$ for these boxes.

More generally, if $\delta_1, \delta_2, \ldots, \delta_n$ are positive real numbers, we have

$$R = R(a; \delta_1, \ldots, \delta_n) := [a_1, a_1+\delta_1) \times \cdots \times [a_n, a_n+\delta_n)$$

$$\bar{R} = \bar{R}(a; \delta_1, \ldots, \delta_n) := [a_1, a_1+\delta_1] \times \cdots \times [a_n, a_n+\delta_n]$$

$$R^\circ = R^\circ(a; \delta_1, \ldots, \delta_n) := (a_1, a_1+\delta_1) \times \cdots \times (a_n, a_n+\delta_n).$$

$R(a; \delta_1, \ldots, \delta_n)$ is called the $(\delta_1, \ldots, \delta_n)$-rectangle with corner a, $\bar{R}(a; \delta_1, \ldots, \delta_n)$ the closed $(\delta_1, \ldots, \delta_n)$-rectangle with corner a, and $R^\circ(a; \delta_1, \ldots, \delta_n)$ the open $(\delta_1, \ldots, \delta_n)$-rectangle with corner a.

We also define the volume of $R, \bar{R}, R^0$ to be

$$\boxed{\mathrm{vol}(R) = \mathrm{vol}(\bar{R}) = \mathrm{vol}(R^0) = \delta_1 \delta_2 \cdots \delta_n .}$$

Boxes are rectangles with all δ_i's equal.

We often call a $(\delta_1, \ldots, \delta_n)$-rectangle a rectangle with sides $\delta_1, \ldots, \delta_n$. A box is a rectangle with all sides equal.

For a δ-box Q,

$$\boxed{\mathrm{vol}(Q) = \delta^n .}$$

Approximations to characteristic functions:

For an interval $I = [a,b] \subset \mathbb{R}$ let

$$K_I^{(m)} : \mathbb{R} \longrightarrow \mathbb{R}, \quad m \in \mathbb{N}$$

and

$$k_I^{(m)} : \mathbb{R} \longrightarrow \mathbb{R}, \quad m \in \mathbb{N}$$

be the functions

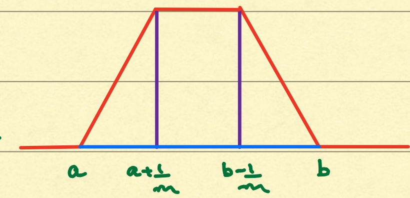
$$K_I^{(m)}(t) = \begin{cases} 1, & a \leq t \leq b \\ mt - am + 1, & a - \frac{1}{m} \leq t < a \\ -mt + bm + 1, & b < t \leq b + \frac{1}{m} \\ 0 & \text{otherwise} \end{cases}$$

and

$$k_I^{(m)}(t) = \begin{cases} 1, & a + \frac{1}{m} \leq t \leq b - \frac{1}{m} \\ mt - ma, & a \leq t < a + \frac{1}{m} \\ -mt + mb, & b - \frac{1}{m} < t \leq b \\ 0 & \text{otherwise} \end{cases}$$

The graphs of $\kappa^{(m)}$ and $k^{(m)}$ are below, showing them to be approximations to $\chi_{[a,b]}$ and $\chi_{(a,b)}$ respectively.

Graph of $\kappa^{(m)}$



Graph of $k^{(m)}$

Now clearly

$$\int_{\mathbb{R}} \kappa_I^{(m)}(t)\,dt = b-a+\frac{1}{m}$$

and

$$\int_{\mathbb{R}} k_I^{(m)}(t)\,dt = b-a-\frac{1}{m}$$

where the integrals are Riemann integrals.

We can get something similar in higher dimensions as follows.

Suppose $a \in \mathbb{R}^n$, say $a = (a_1, \ldots, a_n)$. For $R = R(a; \delta_1, \ldots, \delta_n)$ as in (*) (i.e, $\delta_i > 0$, $i = 1, \ldots, n$) define $I_k = [a_k, a_k + \delta_k]$, $k = 1, \ldots, n$ and set

$$\kappa_R^{(m)}(t_1, \ldots, t_n) = \prod_{j=1}^{n} \kappa_{I_j}^{(m)}(t_j), \quad m \in \mathbb{N}$$

and

$$k_R^{(m)}(t_1, \ldots, t_n) = \prod_{j=1}^{n} k_{I_j}^{(m)}(t_j), \quad m \in \mathbb{N}.$$

Then $\kappa^{(m)} \longrightarrow \chi_{\bar{R}}$ and $k^{(m)} \longrightarrow \chi_{R^\circ}$

as $m \longrightarrow \infty$, where $\bar{R} = \bar{R}(a; \delta_1, \ldots, \delta_n)$ and $R^\circ = R^\circ(a; \delta_1, \ldots, \delta_n)$.

Standard Riemann integration gives us:

$$\Lambda(\chi_R^{(m)}) = \int_{\mathbb{R}^n} \chi_R^{(m)}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n$$

$$= \prod_{j=1}^{n} \int_{\mathbb{R}} \chi_{I_j}^{(m)}(t_j)\, dt_j$$

$$= \prod_{j=1}^{n} \left(\delta_j + \frac{1}{m}\right), \quad m \in \mathbb{N}.$$

Similarly

$$\Lambda(k_R^{(m)}) = \prod_{j=1}^{n} \left(\delta_j - \frac{1}{n}\right), \quad n \in \mathbb{N}.$$

Let $\mathcal{L}$ be the Lebesgue σ-alg on $\mathbb{R}^n$ and m the Lebesgue measure on $\mathbb{R}^n$ and $\Lambda : C_c(\mathbb{R}^n) \longrightarrow \mathbb{C}$ the Riemann integral functional, i.e., $\Lambda f = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n$, $f \in C_c(\mathbb{R}^n)$.

Now

$$m(\bar{R}) = \int_{\mathbb{R}^n} \chi_{\bar{R}}\, dm$$

$$= \lim_{m \to \infty} \int_{\mathbb{R}^n} \chi_R^{(m)}\, dm \qquad (\text{by DCT})$$

$$= \lim_{m \to \infty} \Lambda(\chi_R^{(m)})$$

$$= \delta_1 \cdots \delta_n$$

Similarly $m(R^\circ) = \delta_1 \cdots \delta_n$.