Sep 11, 2018

Lecture 10

The Lebesgue measure on $\mathbb{R}^n$

As mentioned earlier, let

$$\Lambda : C_c(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

be given by

$$\Lambda f = \int_{\mathbb{R}^n} f(t_1, \dots, t_n)\, dt_1 \cdots dt_n \quad , \; f \in C_c(\mathbb{R}^n)$$

where the right side is the Riemann integral of f over a closed rectangle containing Supp$f$. The answer is clearly independent of the rectangle of integration chosen.

Let $(\mathcal{L}, m)$ be the corresponding complete σ-algebra and measure. $\mathcal{L}$ is called the Lebesgue σ-algebra on $\mathbb{R}^n$ and $m : \mathcal{L} \longrightarrow [0, \infty]$ the Lebesgue measure. If we wish to emphasize the role of n, we write $\mathcal{L}_n$ and μ_n for $\mathcal{L}$ and μ.

Since every open set in $\mathbb{R}^n$ is σ-compact, m is clearly regular.

For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, let $\tau_a : C_c(\mathbb{R}^n) \longrightarrow C_c(\mathbb{R}^n)$ be

$$(\tau_a f)(x) = f(x - a) \qquad x \in \mathbb{R}^n.$$

Then

(1) ———
$$\int_{\mathbb{R}^n} (\tau_a f)(t_1, \dots, t_n)\, dt_1 \cdots dt_n = \int_{\mathbb{R}^n} f(t_1, \dots, t_n)\, dt_1 \cdots dt_n .$$

In fact, if $I_1, \dots, I_n$ are closed bounded intervals in $\mathbb{R}$ and $R = I_1 \times \dots \times I_n$, then, as is well-known

$$\int_R f(t_1,\dots,t_n)\,dt_1\cdots dt_n = \int_{R+a} (\tau_a f)(t_1,\dots,t_n)\,dt_1\cdots dt_n .$$

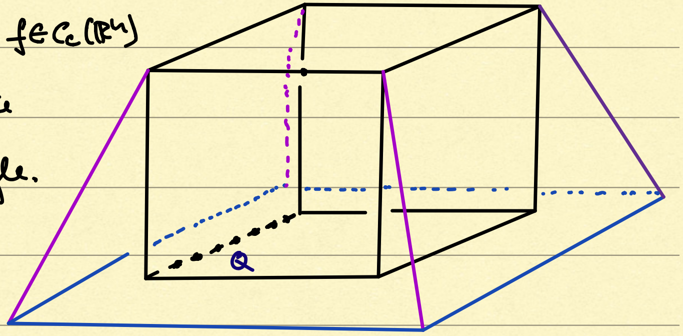
In somewhat greater detail, for $f \in C_c(\mathbb{R}^n)$, K a compact subset of $\mathbb{R}^n$, and V an open subset of $\mathbb{R}^n$ we have

- $\operatorname{Supp} \tau_a f = (\operatorname{Supp} f) + a$
- $\{g \in C_c(\mathbb{R}^n) \mid g \prec V+a\} = \{g = \tau_a h \mid h \in C_c(\mathbb{R}^n),\ h \prec V\}$
- $\{g \in C_c(\mathbb{R}^n) \mid K+a \prec g\} = \{g = \tau_a h \mid h \in C_c(\mathbb{R}^n),\ K \prec h\}$

From the above and (1) and the construction of $(\mathcal{L}, m)$ from Λ, it is easy to see that

$$\boxed{m(E+a) = m(E), \quad E \in \mathcal{L}.}$$

To understand the Lebesgue measure we need to work out measures of δ-boxes and various rectangles. Recall that in the last class we showed that $\mathcal{B}(X)$ has a subset Ω consisting of 2^{-n}-boxes, $n \in \mathbb{N}$, such that if μ, ν are Borel measures s.t., $\mu(Q) = \nu(Q) < \infty$ $\forall\, Q \in \Omega$, then $\mu = \nu$. Working out $m(Q)$ needs us to approximate χ_Q by elements of $C_c(\mathbb{R}^n)$. The picture displayed is of graph of $f \in C_c(\mathbb{R}^n)$ with $\bar{Q} \prec f \prec V$, where V is the open set outside the blue rectangle. As the blue rectangle approaches $\bar{Q}$ $f \longrightarrow \chi_{\bar{Q}}$. ($\bar{Q}$ = closure of Q)



Notations and terminology

Fix $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Recall that a δ-box with corner a, for a positive real number a, is

$$Q(a,\delta) = [a_1, a_1+\delta) \times \dots \times [a_n, a_n+\delta).$$

We also define the closed δ-box with corner a, $\bar{Q}(a,\delta)$ and the open δ-box with corner a, $Q^\circ(a,\delta)$ to be

$$\bar{Q}(a,\delta) = [a_1, a_1+\delta] \times \dots \times [a_n, a_n+\delta]$$

and
$$Q^\circ(a,\delta) = (a_1, a_1+\delta) \times \dots \times (a_n, a_n+\delta).$$

There are of course many sets between $Q^\circ(a,\delta)$ and $\bar{Q}(a,\delta)$ other than $Q(a,\delta)$ but for the moment these three boxes are all we need.

$$Q^\circ(a,\delta) \subset Q(a,\delta) \subset \bar{Q}(a,\delta).$$

If a and δ are understood from the context, we write $Q^\circ, Q, \bar{Q}$ for these boxes.

More generally, if $\delta_1, \delta_2, \dots, \delta_n$ are positive real numbers, we have

$$R = R(a; \delta_1, \dots, \delta_n) := [a_1, a_1+\delta_1) \times \dots \times [a_n, a_n+\delta_n)$$
$$\bar{R} = \bar{R}(a; \delta_1, \dots, \delta_n) := [a_1, a_1+\delta_1] \times \dots \times [a_n, a_n+\delta_n]$$
$$R^\circ = R^\circ(a; \delta_1, \dots, \delta_n) := (a_1, a_1+\delta_1) \times \dots \times (a_n, a_n+\delta_n).$$

$R(a; \delta_1, \dots, \delta_n)$ is called the $(\delta_1, \dots, \delta_n)$-rectangle with corner a, $\bar{R}(a; \delta_1, \dots, \delta_n)$ the closed $(\delta_1, \dots, \delta_n)$-rectangle with corner a, and $R^\circ(a; \delta_1, \dots, \delta_n)$ the open $(\delta_1, \dots, \delta_n)$-rectangle with corner a.

We also define the volume of $R, \bar{R}, R^{\circ}$ to be

$$\boxed{\operatorname{vol}(R) = \operatorname{vol}(\bar{R}) = \operatorname{vol}(R^{\circ}) = \delta_1 \delta_2 \dots \delta_n .}$$

Boxes are rectangles with all δ_i's equal.

We often call a $(\delta_1, \dots, \delta_n)$-rectangle a rectangle with sides $\delta_1, \dots, \delta_n$. A box is a rectangle with all sides equal.

For a δ-box Q,

$$\boxed{\operatorname{vol}(Q) = \delta^n .}$$

Approximations to characteristic functions:

For an interval $I = [a,b] \subset \mathbb{R}$ let

$$K_I^{(m)} : \mathbb{R} \longrightarrow \mathbb{R} \quad , \quad m \in \mathbb{N}$$

and

$$k_I^{(m)} : \mathbb{R} \longrightarrow \mathbb{R} \, , \quad m \in \mathbb{N}$$

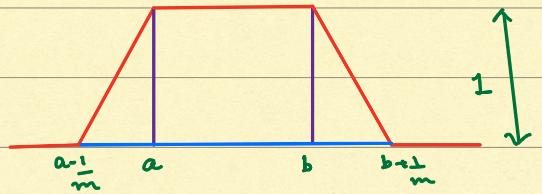
be the functions

$$K_I^{(m)}(t) = \begin{cases} 1 \, , & a \le t \le b \\ mt - am + 1 \, , & a - \frac{1}{m} \le t < a \\ -mt + bm + 1 \, , & b < t \le b + \frac{1}{m} \\ 0 & \text{otherwise} \end{cases}$$

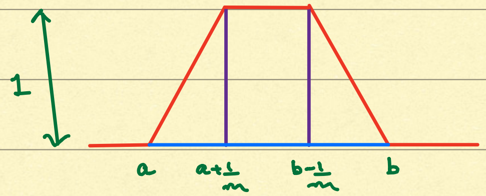
and

$$k_I^{(m)}(t) = \begin{cases} 1 \, , & a + \frac{1}{m} \le t \le b - \frac{1}{m} \\ mt - ma \, , & a \le t < a + \frac{1}{m} \\ -mt + mb \, , & b - \frac{1}{m} < t \le b \\ 0 & \text{otherwise} \end{cases}$$

The graphs of $\mathcal{K}^{(m)}$ and $k^{(m)}$ are below, showing them to be approximations to $\chi_{[a,b]}$ and $\chi_{(a,b)}$ respectively.



Graph of $\mathcal{K}^{(m)}$



Graph of $k^{(m)}$

Now clearly

$$\int_{\mathbb{R}} \mathcal{K}_I^{(m)}(t)\,dt = b-a+\frac{1}{m}$$

and

$$\int_{\mathbb{R}} k_I^{(m)}(t)\,dt = b-a-\frac{1}{m}$$

where the integrals are Riemann integrals.

We can get something similar in higher dimensions as follows.

Suppose $a \in \mathbb{R}^n$, say $a=(a_1,\dots,a_n)$. For $R=R(a;\delta_1,\dots,\delta_n)$ as in (*) (i.e, $\delta_i>0$, $i=1,\dots,n$) define $I_k=[a_k, a_k+\delta_k]$, $k=1,\dots,n$ and set

$$\mathcal{K}_R^{(m)}(t_1,\dots,t_n) = \prod_{j=1}^{n} \mathcal{K}_{I_j}^{(m)}(t_j), \quad m \in \mathbb{N}$$

and

$$k_R^{(m)}(t_1,\dots,t_n) = \prod_{j=1}^{n} k_{I_j}^{(m)}(t_j), \quad m \in \mathbb{N}.$$

Then $\mathcal{K}^{(m)} \longrightarrow \chi_{\bar{R}}$ and $k^{(m)} \longrightarrow \chi_{R^0}$

as $m \longrightarrow \infty$, where $\bar{R} = \bar{R}(a; \delta_1, \ldots, \delta_n)$ and $R^\circ = R^\circ(a; \delta_1, \ldots, \delta_n)$.

Standard Riemann integration gives us:

$$\begin{aligned} \Lambda\left(\chi_R^{(m)}\right) &= \int_{\mathbb{R}^n} \chi_R^{(m)}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n \\ &= \prod_{j=1}^{n} \int_{\mathbb{R}} \chi_{I_j}^{(m)}(t_j)\, dt_j \\ &= \prod_{j=1}^{n} \left(\delta_j + \frac{1}{m}\right), \quad m \in \mathbb{N}. \end{aligned}$$

Similarly

$$\Lambda\left(k_R^{(m)}\right) = \prod_{j=1}^{n} \left(\delta_j - \frac{1}{n}\right), \quad n \in \mathbb{N}.$$

Let $\mathcal{L}$ be the Lebesgue σ-alg on $\mathbb{R}^n$ and m the Lebesgue measure on $\mathbb{R}^n$ and $\Lambda : C_c(\mathbb{R}^n) \longrightarrow \mathbb{C}$ the Riemann integral functional, i.e., $\Lambda f = \int_{\mathbb{R}^n} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n$, $f \in C_c(\mathbb{R}^n)$.

Now

$$\begin{aligned} m(\bar{R}) &= \int_{\mathbb{R}^n} \chi_{\bar{R}}\, dm \\ &= \lim_{m \to \infty} \int_{\mathbb{R}^n} \chi_R^{(m)}\, dm \qquad \text{(by DCT)} \\ &= \lim_{m \to \infty} \Lambda\left(\chi_R^{(m)}\right) \\ &= \delta_1 \cdots \delta_n \end{aligned}$$

Similarly $m(R^\circ) = \delta_1 \cdots \delta_n$.

It follows that, since $R^\circ \subset R \subset \bar{R}$, that

$$\boxed{m(R) = \delta_1 \dots \delta_n = \text{vol}\, R.}$$

In fact for any S s.t. $R^\circ \subset S \subset \bar{R}$, $m(S) = \delta_1 \dots \delta_n$.

In particular the faces of $\bar{R}$ have Lebesgue measure 0.

This in turn means:

Proposition: For $i = 1, \dots, n$ let $S_i = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid a_i = 0\}$. Then $m(S_i) = 0$.

Proof: This is an immediate consequence of the countable additivity of m, and the fact that the faces of $\bar{R}$ have m measure zero. q.e.d.

Translation invariant measures:

As we have seen, m is translation invariant on $\mathcal{L}$.

Suppose μ is translation invariant on $\mathcal{B}(X)$.

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\delta_1, \delta_2, \dots, \delta_n$ be positive real numbers. Let $R = R(a; \delta_1, \dots, \delta_n)$. Fix a positive real number k k, and write $R' = R(a; k\delta_1, \delta_2, \dots, \delta_n)$.

If $k \in \mathbb{N}$, then R' can be written as the disjoint union

$$R' = R \cup (R + e_1) \cup (R + 2e_1) \cup \dots \cup (R + (k-1)e_1)$$

where $e_1 = (1, 0, \dots, 0)$. Since μ is translation invariant, we have

$$\mu(R') = k\,\mu(R)$$

If $k = \frac{1}{\ell}$, then R can be written as the disjoint union of $R' + \frac{j}{\ell} e_1$, $j = 0, \dots, \ell - 1$, whence $\mu(R) = \ell\,\mu(R')$, and hence once again we have

$$\mu(R') = k\mu(R).$$

Combining the two we see

$$\mu(R') = k(\mu) \qquad \text{for } k \in \mathbb{Q} \cap [0,\infty).$$

If $k_m \uparrow k$, $k_m > 0$ $\forall m$, $k > 0$, then clearly the rectangles $R_m = R(a; k_m\delta_1, \delta_2, \dots, \delta_n)$ increase to R'. Since k can be written as the increasing limit of rationals this means

$$\mu(R') = \lim_{m\to\infty} \mu(R_m) = \lim_{m\to\infty} k_m \mu(R) = k\mu(R).$$

A little thought shows that this argument can be repeated with δ_1 replaced by δ_j for any j. Hence we have

$$(*) \quad \boxed{\mu(a;k_1\delta_1,\dots,k_n\delta_n) = k_1\cdots k_n\mu(a;\delta_1,\dots,\delta_n) \quad \left.\begin{array}{l} a\in\mathbb{R}^n \\ k_i>0 \ \ i=1,\dots,n \\ \delta_i>0 \ \ i=1,\dots,n \end{array}\right\}}$$

In view of (*) we have the following

Theorem: Let $Q_0 = Q(0;1)$ and $c = \mu(Q_0)$ where μ is a translation invariant positive measure on $\mathcal{B}(X)$

$$\mu = cm\big|_{\mathcal{B}(X)}$$

where m is the Lebesgue measure on $\mathbb{R}^n$.

Proof: By (*) we see that for a δ-box Q, we have

$\mu(Q) = \delta^n \mu(Q_0) = c\delta^n = c\,m(Q_0)$. The result follows from earlier results (see Proposition towards the end of the previous lecture).

Proposition: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, $\mathcal{L}$ the Lebesgue σ-algebra on $\mathbb{R}^n$ and m the Lebesgue measure on $\mathbb{R}^n$. Let $\mu : \mathcal{L} \longrightarrow [0,\infty]$ be the measure

$$\mu(E) := m(T(E)), \qquad E \in \mathcal{L}.$$

Then there exists a non-negative scalar $\Delta(T)$ such that

$$\mu(E) = \Delta(T)\, m(E).$$

Proof:

Clearly μ is translation invariant. From the previous result, if $\Delta T = \mu(Q_0)$ then $\mu(E) = \Delta(T)\, m(E)$ for $E \in \mathcal{B}(X)$. For a general $E \in \mathcal{L}$, since $\mathbb{R}^n$ is σ-compact, there exists an F_σ set A and a G_δ set B such that $A \subset E \subset B$ and $m(B-A) = 0$. Now A, B are Borel sets and hence $\mu(A) = \Delta(T)\, m(A)$ and $\mu(B) = \Delta(T)\, m(B)$. Thus

$$\Delta(T) m(E) = \Delta(T) m(A) = \mu(A) \le \mu(E) \le \mu(B) = \Delta(T)\, m(B) = \Delta(T)\, m(E).$$

It follows that $\mu(E) = \Delta(T)\, m(E)$.

Remark: It is easy to see that $\Delta(T) = |\det T|$ by seeing this is true for elementary linear transformations and then noting that every linear transformation is the product of elementary linear transformations. In greater detail, it is clear that

$\Delta(T_1T_2) = \Delta(T_1)\,\Delta(T_2)$ for any two linear transformations T_1 and T_2 and $|\det(T_1T_2)| = |\det T_1|\,|\det T_2|$. Now T is the product of the following types of linear transformation S

(a) $\{Se_1, \dots, Se_n\}$ is a permutation of $\{e_1, \dots, e_n\}$

(b) $Se_1 = \alpha e_1$, $Te_i = e_i$, $i = 2, \dots, n$

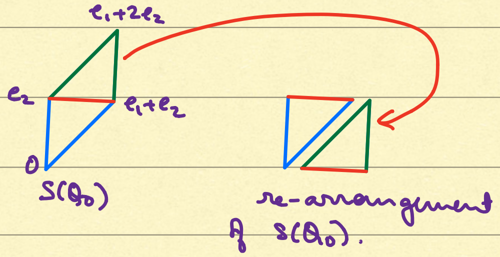
(c) $Se_1 = e_1 + e_2$, $Se_i = e_i$, $e = 2, \dots, n$.

For S as in (a), $S(Q_0) = Q_0$, whence $\Delta(S) = 1$. On the other hand $|\det S|$ is also equal to 1.

For (b) $S(Q_0) = R(0; \alpha, 1, \dots, 1)$ if $\alpha > 0$. Hence $m(S(Q_0)) = \alpha = |\det S|$. If $\alpha = 0$, $S(Q_0)$ lies in $\{(a_1, \dots, a_n) \mid a_1 = 0\}$, whence from an earlier Proposition, $m(S(Q_0))$, i.e., $\Delta(S) = 0$. On the other hand if $\alpha = 0$, clearly $\det S = 0$. If $\alpha < 0$, then $S(Q_0)$ is sandwiched between $(\alpha, 0) \times \overbrace{[0,1) \times \dots \times [0,1)}^{n-1 \text{ factors}}$ and $[\alpha, 0] \times \overbrace{[0,1] \times \dots \times [0,1]}^{n-1 \text{ factors}}$ and hence $m(S(Q_0)) = -\alpha = |\alpha| = |\det S|$. Thus $\Delta(S) = |\det S|$ in this case too.

(c) In this case $S(Q_0)$ is a <u>disjoint union</u>

$$S(Q_0) = A \sqcup (B + e_2)$$



where

$$A = \{(x_1, \dots, x_n) \mid 0 \le x_1 \le x_2,\ 0 \le x_i < 1,\ i = 2, \dots, n\}$$

and

$$B = \left\{(x_1, \dots, x_n) \;\middle|\; \begin{array}{l} 0 \le x_2 < x_1,\ 0 < x_1 < 1 \\ 0 \le x_i < 1,\ i = 3, \dots, n \end{array}\right\}$$

Thus
$$\begin{aligned} m(S(Q_0)) &= m(A) + m(B + e_1) \\ &= m(A) + m(B) \quad \text{(by translation invariance of } m\text{)} \\ &= m(A \cup B) = m(Q_0) = 1. \Rightarrow \Delta(T) = 1 = |\det S|. \end{aligned}$$

This proves the assertion. q.e.d.