

Aug 7, 2018

Lecture 1

Sigma Algebras (σ -algebras)

Let X be a non-empty set.

(a) A collection \mathcal{M} of subsets of X is called a σ -algebra in X (sometimes "on X ") if the following conditions are met:

(i) $X \in \mathcal{M}$

(ii) $\forall A \in \mathcal{M}$, then $A^c \in \mathcal{M}$, where A^c denotes the complement of A in X

(iii) $\forall A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathcal{M}$ for $n=1, 2, 3, \dots$, then $A \in \mathcal{M}$.

(b) \mathcal{M} is a σ -algebra in X then (X, \mathcal{M}) is called a measurable space. Members of \mathcal{M} are called measurable sets or sometimes \mathcal{M} -measurable sets.

(c) Y is a topological space and

$$f: X \longrightarrow Y$$

map. Then f is said to be measurable if $f^{-1}(V)$ is measurable set for every open set V in Y .

Basic Properties: Suppose (X, \mathcal{M}) is a measurable space. Then the following properties hold:

(a) $\emptyset \in \mathcal{M}$. This is so because $\emptyset = X^c$ and $X \in \mathcal{M}$.

(b) $\forall A_1, A_2, \dots, A_n \in \mathcal{M}$ then $A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{M}$. This is seen by setting $A_{n+k} = \emptyset$ for all $k \geq 1$ and

using property (iii) of σ -algebras.

(c) $A_n \in \mathcal{M}$ for $n=1, 2, 3, \dots$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

This is seen by observing that

$$\left(\bigcap_{n=1}^{\infty} A_n \right)^c = \bigcup_{n=1}^{\infty} A_n^c.$$

Since \mathcal{M} is closed under formation of countable unions and taking complements, the assertion follows.

(d) $A, B \in \mathcal{M}$, then $A - B \in \mathcal{M}$ where

$$A - B = \{x \in X \mid x \in A \text{ and } x \notin B\}.$$

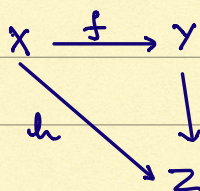
since $A - B = A \cap B^c$, the assertion is clear.

Note: The prefix σ in σ -algebras refers to the fact that property (iii) of σ -algebras is required to hold for all countable unions of members of \mathcal{M} . If (iii) is required for finite unions only, then \mathcal{M} is called an algebra of sets.

Theorem: Let Y, Z be topological spaces and $g: Y \rightarrow Z$ a continuous map.

(a) If X is a topological space and $f: X \rightarrow Y$ a continuous function, then $h = g \circ f$ is continuous.

(b) If X is a measurable space and $f: X \rightarrow Y$ a measurable map, then $g \circ f$ is measurable.



$$(h = g \circ f, h: X \rightarrow Z).$$

Proof: Let V be an open set in Z . For both parts (a) and (b) we have

$$h^{-1}(V) = f^{-1}(g^{-1}(V)). \quad \text{—————} (*)$$

For part (a), since f and g are continuous, $g^{-1}(V)$ and $f^{-1}(g^{-1}(V))$ are both open, proving h is continuous via (*).

For part (b), since g is continuous $g^{-1}(V)$ is open in Y , and since f is measurable, $f^{-1}g^{-1}(V)$ is measurable in X . By (*) we are done.

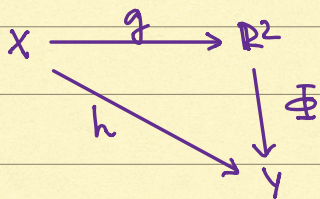
Theorem: Let u and v be measurable functions on a measurable space X , and let $\Phi: \mathbb{R}^2 \rightarrow Y$ be a continuous map from the plane \mathbb{R}^2 into topological space Y . Define

$$h(x) = \Phi(u(x), v(x))$$

for $x \in X$. Then $h: X \rightarrow Y$ is measurable.

Proof:

Let $g: X \rightarrow \mathbb{R}^2$ be the map which sends a point x in X to the point $(u(x), v(x))$ in \mathbb{R}^2 . We have a commutative diagram



Now if A, B are subsets of \mathbb{R} , then clearly

$$g^{-1}(A \times B) = u^{-1}(A) \cap v^{-1}(B).$$

In particular, if \mathcal{M} is the σ algebra underlying the measure space X , and I, J are intervals in \mathbb{R} , then $g^{-1}(I \times J) = u^{-1}(I) \cap v^{-1}(J)$ and the right side lies in \mathcal{M} since $u^{-1}(I) \in \mathcal{M}$ and $v^{-1}(J) \in \mathcal{M}$.

Now the collection \mathcal{C} of open rectangles $I \times J$ in \mathbb{R}^2 with I and J having rational endpoints is a countable collection. Let $V \subseteq \mathbb{R}^2$ be an open set. If $x \in V$, choose a member V_x of \mathcal{C} such that $x \in V_x \subseteq V$. Clearly

$$V = \bigcup_{x \in V} V_x.$$

Now the collection $\{V_x\}$ is countable since \mathcal{C} is countable, and hence can be re-labelled as $\{V_1, V_2, \dots, V_n, \dots\}$. Thus

$$g^{-1}(V) = g^{-1}\left(\bigcup_{n=1}^{\infty} V_n\right) = \bigcup_{n=1}^{\infty} g^{-1}(V_n).$$

Since V_n is an open rectangle, $g^{-1}(V_n) \in \mathcal{M}$ for $n \in \mathbb{N}$.

Hence $g^{-1}(V) \in \mathcal{M}$, since \mathcal{M} is closed under countable unions.

q.e.d.

More Properties: Let (X, \mathcal{M}) be a measure space.

(a) If $f = u + iv$, where u and v are measurable functions on X , then $f: X \rightarrow \mathbb{C}$ is measurable.

Pf: Identify \mathbb{C} with \mathbb{R}^2 . Let \mathcal{C} , and $\mathbb{F}: \mathbb{R}^2 \rightarrow \mathbb{C}$ the identity map. The theorem above applies to this situation.

(b) If $f = u + iv$ is a complex function with $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, then $u, v, |f|$ are real measurable functions.

functions on X .

Proof: Let \mathbb{F} be respectively $z \mapsto \operatorname{Re}(z)$, $z \mapsto \operatorname{Im}(z)$, $z \mapsto |z|$. Applying the previous theorem to each of these functions, the result follows.

(c) If f, g are complex measurable on X , then so are $f+g$ and fg .

Proof: If f and g are real, then the result follows from the previous theorem with $\mathbb{F}(s, t) = s + t$ and $\mathbb{F}(s, t) = st$ respectively. The complex case follows from (a) and (b).

(d) If E is a measurable set in X , and if

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then χ_E is a measurable set.

Proof: For any subset A of \mathbb{R} , $\chi_E^{-1}(A)$ is \emptyset, E, E^c, X (why?) and each of these is measurable.

(e) If f is a complex measurable function on X , then $|f|$ is a complex measurable function α on X such that $|f| = \alpha$ and $f = \alpha^{-1} |f|$.

Proof: Let $E = \{x \in X \mid f(x) = 0\}$. Now $D = f^{-1}(\mathbb{C} - \{0\})$ is measurable since $\mathbb{C} - \{0\}$ is open in \mathbb{C} and f is measurable. It follows that $E = f^{-1}(\{0\}) = D^c$ is measurable. By (d) this means χ_E is measurable. Define $\phi: \mathbb{C} - \{0\} \rightarrow \mathbb{C}$ by

$$\phi(z) = \frac{z}{|z|}$$

Then φ is continuous. Define

$$\alpha = \varphi(f + \chi_E).$$

Note this makes sense for $f(x) + \chi_E(x)$ is never zero for any $x \in X$. Now $f + \chi_E$ is measurable by (c), since f and χ_E are. Further φ is continuous. Since continuous functions of measurable functions are measurable, it follows that α is measurable. Note that

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in E \\ \frac{f(x)}{|f(x)|} & \text{if } x \notin E. \end{cases}$$

Thus $|\alpha| = 1$. Clearly

$$f = \alpha \cdot |f|.$$

q.e.d.