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Lecture 1

Sigma Algebras (σ-algebras)

Let X be a non-empty set.

(a) A collection $\mathcal{M}$ of subsets of X is called a σ-algebra in X (sometimes "on X") if the following conditions are met:

(i) $X \in \mathcal{M}$

(ii) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$, where A^c denotes the complement of A in X

(iii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathcal{M}$ for $n = 1, 2, 3, \ldots$, then $A \in \mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra in X then $(X, \mathcal{M})$ is called a measurable ace. Members of $\mathcal{M}$ are called measurable sets or sometimes $\mathcal{M}$-measurable sets.

(c) Y is a topological space and

$$f : X \longrightarrow Y$$

map. Then f is said to be measurable if $f^{-1}(V)$ is measurable set for every open set V in Y.

Basic Properties: Suppose $(X, \mathcal{M})$ is a measurable space. Then the followi properties hold:

(a) $\phi \in \mathcal{M}$. This is so because $\phi = X^c$ and $X \in \mathcal{M}$.

(b) If $A_1, A_2, \ldots, A_n \in \mathcal{M}$ then $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{M}$. This is seen by setting $A_{n+k} = \phi$ for all $k \geq 1$ and using property (iii) of σ-algebras.

(c) $A_n \in \mathcal{M}$ for $n = 1, 2, 3, \dots$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

This is seen by observing that

$$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c.$$

since $\mathcal{M}$ is closed under formation of countable unions and taking complements, the assertion follows.

(d) $A, B \in \mathcal{M}$, then $A - B \in \mathcal{M}$ where

$$A - B = \{x \in X \mid x \in A \text{ and } x \notin B\}.$$

since $A - B = A \cap B^c$, the assertion is clear.

Note: The prefix σ in σ-algebras refers to the fact that property (iii) of σ-algebras is required to hold for all countable unions of members of $\mathcal{M}$. If (iii) is required for finite unions only, then $\mathcal{M}$ is called an algebra of sets.

Theorem: Let Y Z be topological spaces and $g: Y \longrightarrow Z$ a continuous map.

(a) If X is a topological space and $f: X \longrightarrow Y$ a continuous function, then $h = g \circ f$ is continuous

(b) If X is a measurable space and $f: X \longrightarrow Y$ a measurable map, then $g \circ f$ is measurable.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d] \\ & Z \end{tikzcd} \qquad (h = g \circ f,\ h: X \longrightarrow Z).$$

Proof : Let V be an open set in Z. For both parts (a) and (b) we have

$$h^{-1}(V) = f^{-1}(g^{-1}(V)). \quad \text{———————} \quad (*)$$

For part (a), since and g are continuous, $g^{-1}(V)$ and $f^{-1}(g^{-1}(V))$ are both , proving h is continuous via $(*)$. For part (b), since g · continuous $g^{-1}(V)$ is open in Y, and since f is measur , $f^{-1}g^{-1}(V)$ is measurable in X. By $(*)$ we are done.

Theorem : Let u and v be measurable functions on a measurable space X, and let $\Phi : \mathbb{R}^2 \longrightarrow Y$ be a continuous map from the plane $\mathbb{R}^2$ into topological space Y. Define

$$h(x) = \Phi(u(x \quad v(x))$$

for $x \in X$. Then $h : X \longrightarrow$ is measurable.

Proof :

Let $g : X \longrightarrow \mathbb{R}^2$ be the which sends a point x in X to the point $(u(x), v(x))$ in 1^2. We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "h"'] & \mathbb{R}^2 \arrow[d, "\Phi"] \\ & Y \end{tikzcd}$$

Now if A, B are subsets of , then clearly

$$g^{-1}(A \times B) = u^{-1}(A) \quad v^{-1}(B).$$

In particular, if $\mathcal{M}$ is the σ elra underlying the measure space X, and I, J are intervals in $\mathbb{R}$, then $g^{-1}(I \times J) = u^{-1}(I) \cap v^{-1}(J)$ and the right side lies in $\mathcal{M}$ since $u^{-1}(I) \in \mathcal{M}$ $v^{-1}(J) \in \mathcal{M}$.

Now the collection $\mathcal{C}$ of open etangles $I \times J$ in $\mathbb{R}^2$ with I and J having rational points is a countable collection. Let $V \subseteq \mathbb{R}^2$ be an set. If $x \in V$, choose a member V_x of $\mathcal{C}$ such that $V_x \subseteq V$. Clearly

$$V = \bigcup_{x \in V} V_x .$$

Now the collection $\{V_x\}$ is count since $\mathcal{C}$ is countable, and hence can be re-labelled as $\{V_1\ {}_2, \dots, V_n, \dots\}$. Thus

$$g^{-1}(V) = g^{-1}\Big(\bigcup_{n=1}^{\infty} V_n\Big) = \bigcup_{n=1}^{\infty} g^{-1}(V_n .$$

Since V_n is an open rectangle, $g^{-1}(\ {}_n) \in \mathcal{M}$ $\forall\, n \in \mathbb{N}$. Hence $g^{-1}(V) \in \mathcal{M}$, since $\mathcal{M}$ is under countable unions.

q.e.d.

<u>More Properties</u> : Let $(X, \mathcal{M})$ be a measu space.

(a) If $f = u + iv$, where u and v are measurable functions on X, then $f : X \longrightarrow \mathbb{C}$ is measurable.

<u>Pf</u>: Identify $\mathbb{C}$ with $\mathbb{R}^2$. Let $\mathbb{C}$, and $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{C}$ the identity map. the theorem above to this situation.

(b) If $f = u + iv$ is a complex function with $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, then $u, v, |f|$ are real measurable functions on X.

Proof: Let Φ be respectively $z \mapsto$ (z), $z \mapsto \mathrm{Im}(z)$, $z \mapsto |z|$. Applying the previous theorem to each of these functions, the result follows.

(c) If f, g are complex measurable on , then so are $f+g$ and fg.

Proof: If f and g are real, then the follows from the previous theorem with $\Phi(s,t) = s$ and $\Phi(s,t) = st$ respectively. The complex case from (a) and (b).

(d) If E is a measurable set in X, and if

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then χ_E is a measurable set.

Proof: For any subset A of $\mathbb{R}$, $\chi_E^{-1}(A)$ is of ϕ, E, E^c, X (why?) and each of these is -ble.

(e) If f is a complex measurable function on X, is a complex measurable function α on X such $|\alpha| = 1$ and $f = \alpha |f|$.

Proof: Let $E = \{x \in X \mid f(x) = 0\}$. Now $D = f^{-1}(\mathbb{C} - \{0\})$ is measurable since $\mathbb{C} - \{0\}$ is open in $\mathbb{C}$ and f is measurable. It follows that $E = f^{-1}(\{0\}) = D^c$ is measurable. By (d) this means χ_E is measurable. Define $\phi : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$ by

$$\phi(z) = \frac{z}{|z|}$$

Then φ is continuous. Define

$$\alpha = \varphi(f + \chi_E).$$

Note this makes sense for $f(x) + \chi_E(x)$ is never zero for any $x \in X$. Now $f + \chi_E$ is measurable by (c), since f and χ_E are. Further φ is continuous. Since continuous functions of measurable functions are measurable, it follows that α is measurable. Note that

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in E \\ \dfrac{f(x)}{|f(x)|} & \text{if } x \notin E. \end{cases}$$

Thus $|\alpha| = 1$. Clearly

$$f = \alpha \cdot |f|.$$

q.e.d.