HW 9

Convolution of functions.

(1) For $f, g \in L^1(\mathbb{R})$, show that $f * g = g * f$.

(2) Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbb{R})$, and $g \in L^p(\mathbb{R})$. Imitate the proof given in class to shaow that the integral defining $(f * g)(x)$ exists for almost all x, and that $f * g \in L^p(\mathbb{R})$.

Convolution of measures. In the following set of problems, M will denote the Banach space of all complex Borel measures on $\mathbb{R}$. The norm on M is $\|\mu\| = |\mu|(\mathbb{R})$. Let $\mathscr{B} = \mathscr{B}(\mathbb{R})$ and $\mathscr{L}$ the Lebesgue σ-algebra on $\mathbb{R}$ and m the Lebesgue measure on $\mathscr{B}$ and on $\mathscr{L}$. For each $E \in \mathscr{B}$ define

$$E_2 = \{(x, y) \mid x + y = E\} \in \mathbb{R}^2.$$

If μ and λ are in M, define their convolution to be the set functions given by

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_2)$$

for every $E \in \mathscr{B}$. Fix μ and ν in M in what follows. T

(3) Prove that $\mu * \lambda \in M$ and that $\|\mu * \lambda\| \leq \|\mu\| \, \|\lambda\|$.

(4) Prove that $\mu * \lambda$ is the unique $\nu \in M$ such that

$$\int_{\mathbb{R}} f \, d\nu = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x + y) \, d\mu(x) \, d\lambda(y)$$

for every $f \in C_0(\mathbb{R})$.

(5) Prove that convolution in M is commutative, associative, and distributive with respect to addition.

(6) Prove the formula

$$(\mu * \lambda)(E) = \int_{\mathbb{R}} \mu(E - t) \, d\lambda(t)$$

for every $E \in \mathscr{B}$. Here

$$E - t = \{x - t \mid x \in E\}.$$

(7) Define a measure $\nu \in M$ to be *discrete* if ν is concentrated on a countable set; define ν to be *continuous* if $\nu(\{x\}) = 0$ for every $x \in \mathbb{R}$. Show that if μ and λ are discrete, then $\mu * \lambda$ is discrete. Show that if μ is continuous then $\mu * \lambda$ is continuous and show that $\mu * \lambda \ll m$ if $\mu \ll m$.

(8) If $d\mu = f \, dm$ and $d\lambda = g \, dm$ where $f, g \in L^1(\mathbb{R})$, show that $d(\mu * \lambda) = (f * g) dm$.