HW 8

The Hilbert space $L^2(T)$. You may need the results in the notes of Lecture 18 for this section. There is a slight discrepancy between the statement of Theorem 1.1.1 in the notes and what was proven in class but the discrepancy is minor. For all doubts refer to the posted notes rather than the notes you took in class.

(1) Let A be a measurable subset of $[0, 2\pi]$. Show that

$$\lim_{n\to\infty} \int_A \cos nx \, dx = \lim_{n\to\infty} \int_A \sin nx \, dx = 0$$

where $\int_A f(x)dx$ denotes the integral of a measurable function f over A with respect to the Lebesgue measure.

(2) Suppose H is a separable infinite dimensional Hilbert space. Show that H is isomorphic to ℓ^2. [*Hint:* You may use the following fact which you probably know for finite dimensional vector spaces, but is true for infinite dimensional spaces too, namely, if S a subset of a vector space V such that V equals the linear span of S, then one can extract a basis of V from S. And if you are curious, prove this privately using Zorn's Lemma.]

(3) Find a closed subset A of $L^2(T)$, $A \neq \emptyset$, such that $\|\cdot\|$ does not achieve a minimum on A.

Monotone classes and algebras. Please refer to HW 7 for definitions and notations.

(4) Let X be a non-empty set and $\mathscr{A}$ an algebra on X. Let μ and ν be measures on $\sigma(\mathscr{A})$ such that $\mu(A) = \nu(A)$ for every $A \in \mathscr{A}$. Suppose further that there exist $E_n \in \mathscr{A}$, $n \in \mathbb{N}$ such that $X = \bigcup E_n$, and such that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Show that $\mu = \nu$.