HW 7

The usual instructions about margins apply. All vector spaces are over $\mathbb{C}$ (though most results are true over $\mathbb{R}$ too). So if we talk about a normed linear space, or an inner product space, the assumption is that the underlying field is $\mathbb{C}$.

Banach spaces.

(1) Let X be a Banach space. For $x \in X$ let $J(x)\colon X^* \to \mathbb{C}$ be the map $J(x)(\Lambda) = \Lambda(x)$. Show that $J(x)$ is a bounded linear functional on X^*. Show that the resulting map $J\colon X \to X^{**}$ is an isometric isomorphism *into* X. (It need not be onto but you don't have to prove that.)

(2) A Banach space is *separable* if it has a countable dense subset. Prove that ℓ^p is separable for $1 \le p < \infty$, but that ℓ^∞ is not.

Monotone classes and algebras. Let X be a set. A collection of subsets $\mathfrak{M} \subset \mathscr{P}(X)$ is said to be *monotone class* if it is closed under countable monotone unions or intersections, i.e., if $A_n, B_n \in \mathfrak{M}$ ($n \in \mathbb{N}$) are such that

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \ldots, \quad B_1 \supset B_2 \supset \cdots \supset B_n \supset \ldots,$$

then

$$\bigcup_n A \in \mathfrak{M}, \quad \bigcap_n B_n \in \mathfrak{M}.$$

A collection $\mathcal{A}$ of subsets of X is called an *algebra* if it is closed under pairwise union and under complementation. Note that if $\mathcal{A}$ is an algebra, X and $\emptyset$ are in $\mathcal{A}$ and $\mathcal{A}$ is closed under pairwise intersection.

(3) Let $\mathcal{A}$ be an algebra of subsets of X and $\mathfrak{M}$ the smallest monotone class containing $\mathcal{A}$. Show that $\mathfrak{M} = \sigma(\mathcal{A})$, where $\sigma(\mathcal{A})$ is the σ-algebra generated by $\mathcal{A}$.

(4) Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be measurable spaces. A *measurable rectangle* on this data is a subset of $X \times Y$ of the form $A \times B$ with $A \in \mathscr{S}$ and $B \in \mathscr{T}$. Let $\mathscr{R} = \mathscr{R}(\mathscr{S},\, \mathscr{T})$ be the collection of finite unions of measurable rectangles. Show that $\mathscr{R}$ is an algebra.

(5) With notations as above, show that every element of $\mathscr{R}$ can be written as a finite disjoint union of measurable rectangles.

Finitely additive measures. Let $\Sigma = \mathscr{P}(\mathbb{N})$. Let S be the space of all finitely additive, complex valued, set functions on Σ that are bounded: that is, μ in S means

(i) $\mu(\emptyset) = 0$,

(ii) $\sup\{|\mu(E)| : E \subset \mathbb{N}\} < \infty$,

(iii) $\mu(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$ whenever $E_1, \cdots, E_n$ are disjoint elements of Σ.

S is a linear space with the operations

$$(\alpha_1\mu_1 + \alpha_2\mu_2)(E) = \alpha_1\mu_1(E) + \alpha_2\mu_2(E)$$

for all $\mu_1, \mu_2 \in S$, all complex numbers α_1, α_2, and $E \in \Sigma$.

(6) Prove that for $\mu \in S$, the number

$$\|\mu\| = \sup\{\sum_{i=1}^{n} |\mu(E_i)| : \mathbb{N} = \cup_{i=1}^{n} E_i; E_i \text{ disjoint}\}$$

is finite, and that this norm makes S into a Banach space.

(7) Prove that S is isometrically isomorphic to $(\ell^\infty)^*$.

(8) Let J be the canonical isometry of ℓ^1 into $(\ell^\infty)^*$ (see Problem 1). Prove that $\mu \in J(\ell^1)$ if and only if μ is countably additive.