HW 6

The usual instructions about margins apply. All vector spaces are over $\mathbb{C}$ (though most results are true over $\mathbb{R}$ too). So if we talk about a normed linear space, or an inner product space, the assumption is that the underlying field is $\mathbb{C}$.

ℓ^p spaces. As in the mid-term, for $p \geq 1$, $(\ell^p, \|\cdot\|_p)$ will denote $(L^p(\#), \|\cdot\|_p)$, where $\#$ is the counting measure on $(\mathbb{N}, \mathscr{P}(\mathbb{N}))$. This means that ℓ^∞ is the Banach space of bounded sequences $s = \{\xi_n\}$ with $\|s\|_\infty = \sup_n |\xi_n|$. For $1 \leq p < \infty$, ℓ^p is the Banach space of sequences $s = \{\xi_n\}$ such that $\sum_n |\xi_n|^p < \infty$, and $\|s\|_p = \left\{\sum_n |\xi_n|^p\right\}^{\frac{1}{p}}$.

(1) Let c and c_0 be the subspaces of ℓ^∞ consisting of convergent sequences and sequences converging to 0 respectively. Show that c and c_0 are closed subspaces of ℓ^∞.

(2) Show that c_0^* can be identified with ℓ^1 in the following way. Let $a = \{a_n\} \in \ell^1$ and $s = \{\xi_n\} \in c_0$, and consider the sum $\Phi_a(s) = \sum_n a_n \xi_n$. Then show that the association $a \mapsto \Phi_a$ identifies ℓ^1 with c_0^*. [You have to show $\Phi_a(s)$ is defined as a complex number for each $a \in \ell^1$ and $s \in c_0$, that Φ_a is a bounded linear functional on c_0, that $a \mapsto \Phi_a$ is norm preserving, one-to-one, and onto. Onto-ness in this case means that given a bounded linear functional Φ on c_0, there exists $a \in \ell^1$ such that $\Phi = \Phi_a$.]

(3) Show that c^* can be identified with ℓ^1 (i.e., there is an isometric isomorphism between c^* and ℓ^1).

(4) Since c_0 is a closed subspace of c, every bounded linear functional on c gives (by restriction) a bounded linear functional on c_0. We thus have a map $c^* \to c_0^*$ which is clearly linear (this is the transpose map to the inclusion map). Using the identifications in the above two problems, this gives us a map $P\colon \ell^1 \to \ell^1$. Describe P.