HW 5

The usual instructions about margins apply. All vector spaces are over $\mathbb{C}$ (though most results are true over $\mathbb{R}$ too). So if we talk about a normed linear space, or an inner product space, the assumption is that the underlying field is $\mathbb{C}$.

Bounded linear operators. Let $T\colon X \to Y$ be a linear operator between normed linear space. T is said to be *bounded* if there is a finite constant $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all x in X. It is easy to see that T is bounded if and only if it is *continuous*. This will be discussed later in the course. If T is bounded, its *norm* $\|T\|$ is defined to be:

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

It is quite easy to see that $\|T\|$ can also be described in three other ways, namely

- $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$,
- $\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in V \setminus \{0\}\right\}$,
- $\|T\| = \inf M$, where M ranges over real numbers such that $\|Tx\| \leq M\|x\|$.

Bounded linear operators from X to Y with the above norm form a normed linear space denoted $B(X, Y)$.

(1) Show that if Y is a Banach space, then so is $B(X, Y)$.

(2) Let Z be a compact Hausdorff space and μ a Borel measure on Z such that $\mu(Z) < \infty$. Regard $(C(Z)$ as a normed linear space in the usual way, namely via $\|\cdot\|_\infty\colon C(Z) \to [0, \infty)$, where $\|f\|_\infty = \max_{z\in Z} |f(z)|$. Show that $\Lambda\colon C(Z) \to \mathbb{C}$ given by $\Lambda f = \int_Z f d\mu$ is a bounded linear operator. Calculate $\|\Lambda\|$.

Inner Product Spaces and Hilbert Spaces. Let V be an inner product space. We say that x and y in V are *orthogonal* if $\langle x, y\rangle = 0$. The zero vector is orthogonal to all vectors in V. We say $x \in V$ is orthogonal to a subspace M of V if $\langle x, m\rangle = 0$ for every $m \in M$. If an $x$and y are orthogonal we write $x \perp y$. If x is orthogonal to a subspaces M of V, we write $x \perp M$. $M^\perp$ denotes the set of all vectors in V orthogonal to M.

In what follows you may assume the *Cauchy-Schwarz inequality*:

$$|\langle x, y\rangle| \leq \|x\|\|y\| \qquad (x, y \in V)$$

as well as other relations you have seen in past homework assignments and quizzes, for example the *Parallelogram Law*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

and the *Polarization Identity*

$$\langle x, y\rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right).$$

(3) Let $x \in V$. Show that the linear functional $\varphi_x \colon V \to \mathbb{C}$ given by $y \mapsto \langle y, x \rangle$ is a bounded linear functional and $\|\varphi_x\| = \|x\|$.

(4) Let M be a subspace of V. (Note that M is an inner product space in its own right.)
 (a) Show that $M^\perp$ is also a subspace of V. Is it a closed subspace of V?
 (b) Show that if V is a Hilbert space and M a closed subspace, then $M^\perp$ is closed and $V = M \bigoplus M^\perp$.

(5) Let x be a vector in V and let M be a subspace of V.
 (a) Show that a point m_0 in M is closest to x amongst all points m in M is and only if $x - m_0 \in M^\perp$. (Note: There may no "closest point to x" in M, in which case the "if and only if" statement says that there is no $m_0 \in M$ such that $x - m_0 \in M^\perp$.)
 (b) Suppose m, m' are elements of M such that $x - m$ and $x - m'$ lie in $M^\perp$. Show that $m = m'$. Conclude that there is at most one point m_0 in M such that $\|x - m_0\| = \inf_{m \in M} \|x - m\|$.

Now suppose M is a complete subspace of V, i.e., suppose M is a subspace which is a Hilbert space (even if V is not). Then one can show (and we will in class when we do Functional Analysis) that for every x in V there exists $Px \in M$ such that $x - Px \in M^\perp$. Let us assume this fact in the exercises which follow. The assignment $x \mapsto Px$ gives us a map $P \colon V \to M$.

(6) With $P \colon V \to M$ as above, show that $P \colon V \to M$ is a bounded linear transformation and $\|P\| = 1$ if $M \neq 0$.

(7) (Riesz Representation for Hilbert spaces) Suppose V is a Hilbert space and $\lambda \colon V \to \mathbb{C}$ is a bounded linear functional.
 (a) Show that $(\ker \lambda)^\perp$ is one dimensional if λ is non-zero.
 (b) Show that there exists a unique element $x_\lambda \in V$ such that $\lambda y = \langle y, x \rangle$ for all $y \in V$. [Hint: Look for x_λ in the one dimensional space in part (a) above.]
 (c) Is $\lambda \mapsto x_\lambda$ a linear transformation from $B(V, \mathbb{C})$ to V?

Measure Theory. In what follows, $(X, \mathscr{M}, \mu)$ is a measure space. For $p \in [1, \infty)$, $L^p(\mu)$ is the space of equivalence classes of functions $f \colon X \to \mathbb{C}$ with $\int_X |f|^p d\mu < \infty$, the equivalence being $f \sim g$ if $f = g$ a.e. $[\mu]$. The norm $\|\cdot\|_p$ on $L^p(\mu)$ is $\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$. With this norm $L^p(\mu)$ is a Banach space (this will be proven in class). The $p = 2$ case is interesting because $L^2(\mu)$ is a *Hilbert Space*, the inner product being $\langle f, g \rangle = \int_X f\bar{g} d\mu$ for $f, g \in L^2(\mu)$. Clearly the Hilbert space norm agrees with the $L^2(\mu)$ norm just described. In what follows assume the facts we stated above. As usual, feel free to use Cauchy-Schwarz.

(8) Suppose μ is finite (i.e., $\mu(X) < \infty$).
 (a) Show that if f is bounded then $f \in L^p(\mu)$ for all $p \geq 1$.
 (b) Show that if $f \in L^2(\mu)$ then $f \in L^1(\mu)$.

(9) Let $f \in L^1(\mu)$, $f \geq 0$, and let ν be the measure on $\mathscr{M}$ given by

$$\nu(E) = \int_E f d\mu \qquad (E \in \mathscr{M}).$$

Suppose $\mu(E) = 0$ whenever $\nu(E) = 0$ (the reverse implication is clearly true).

(a) Show that $\mu(\{x \mid f(x) = 0\}) = 0$ so that f^{-1} is well-defined a.e. $[\mu]$ as well as a.e. $[\nu]$.

(b) Show that if $g \in L^1(\mu)$ then $gf^{-1} \in L^1(\nu)$ and

$$\int_X g d\mu = \int_X g f^{-1} d\nu.$$

(10) Suppose μ is finite on $\mathscr{M}$ and ν is a measure on $\mathscr{M}$ such that $\nu \leq \mu$.

(a) Show that if $f \in L^p(\mu)$ then $f \in L^p(\nu)$ for $p \in [1, \infty)$.

(b) Show that $f \mapsto \int_X f d\nu$ is a bounded linear functional on $L^p(\mu)$.

(c) Show that there exists a function $f \in L^1(\mu)$, $f \geq 0$, such that

$$\nu(E) = \int_E f d\mu \qquad (E \in \mathscr{M}).$$