HW 4

General comments: To help the TA mark the HW, please leave margins on the left, and space between each answer. These spaces are often used by markers to make comments (short ones in the margin, and more general ones after your answer).

Inner Product Spaces. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and V a vector space over $\mathbb{K}$. Recall that an *inner product on* V is a map

$$\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{K}$$

such that for x, y, and z in V and $\alpha \in \mathbb{K}$ the following conditions are satisfied.

- Positive Definiteness:
$$\langle x, x \rangle \geq 0$$
$$\langle x, x \rangle = 0 \Leftrightarrow x = 0$$
- Conjugate linearity:
$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$
- Linearity in the first argument:
$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$
$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

An *inner product space* over $\mathbb{K}$ is a vector space over $\mathbb{K}$ with an inner product. If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, the map

$$\| \cdot \| \colon V \to \mathbb{R}$$

given by

$$\|x\| = \sqrt{\langle x, x \rangle} \qquad (x \in V)$$

is a norm on V. If the inner product space V is a Banach space with respect to this norm, it is called a *Hilbert Space*. In what follows you may assume the *Cauchy-Schwarz inequality*:

$$|\langle x, y \rangle| \leq \|x\| \|y\| \qquad (x, y \in V).$$

(1) Let V be an inner product space. Show that the map $\| \cdot \|$ defined above is a norm on V.

(2) (Geometry in Inner Product Spaces)
   (a) (*Parallelogram Law*) Show that in any inner product space
$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(b) (*Polarization Identity*) Show that in any inner product space over $\mathbb{C}$

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

which expresses the inner product in terms of the norm.

(c) Use (a) or (b) to show that the norm on $C([0,1])$ does not come from an inner product. Recall that "the norm" on $C([0,1])$ is

$$\|\cdot\|_\infty \colon C([0,1]) \to [0,\,\infty)$$

given by $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$, $f \in C([0,1])$.

Functions of Bounded Variation. There is a natural generalisation of the notion of monotone functions on an interval I, viz., the notion of functions of *bounded variation.*

Definition: Let $\mathbf{D} : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $I = [a,\, b]$. For a function $f : I \to \mathbb{R}$ define

$$V(f,\, \mathbf{D}) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$
$$V(f,\, I) = \sup_{\mathbf{D}} V(f,\, \mathbf{D}).$$

The function f is said to be of *bounded variation on* I if $V(f,\, I) < \infty$. We denote the space of functions of bounded variation on I by $BV(I)$.

(3) Show that $BV(I)$ is a vector space over $\mathbb{R}$ with the obvious notion of addition and scalar multiplication.

(4) Let $J_1 = [a,\, b]$ and $J_2 = [b,\, c]$ and $J = [a,\, c]$. Show that $V(f,\, J) = V(f,\, J_1) + V(f,\, J_2)$ for every function $f : J \to \mathbb{R}$.

Lebesgue measure on $\mathbb{R}$. Let $\Lambda\colon C_c(\mathbb{R}) \to \mathbb{C}$ be given by $\Lambda f = \int_{-\infty}^{\infty} f(t)dt$, where the right side is the Riemann integral over any closed interval containing $\mathrm{Supp}(f)$. Λ is clearly a positive linear functional on $C_c(\mathbb{R})$. According to what we have been doing in class (Riesz Representation Theorem, or Stone's Theorem), we get a σ-algebra $\mathscr{L}$ on $\mathbb{R}$ containing $\mathscr{B}(X)$ and a measure $m\colon \mathscr{L} \to [0,\infty]$ which is finite on compact sets, complete ... (read the properties) such that $\Lambda f = \int_{\mathbb{R}} f dm$ for $f \in C_c(\mathbb{R})$. The σ-algebra $\mathscr{L}$ is called the *Lebesgue σ-algebra* on $\mathbb{R}$, and m is called the *Lebesgue measure* on $\mathbb{R}$. In what follows you may assume that finite and countable sets $E \subset \mathbb{R}$ are null (i.e. $m(E) = 0$), that $m(a,b) = m([a,b)) = m((a,b]) = m([a,b]) = b - a$ for $a < b$, $a,b \in \mathbb{R}$, and that m is "translation invariant". i.e., $m(E+a) = m(E)$, $E \in \mathscr{L}$ and $a \in \mathbb{R}$.

(5) Suppose $\{s_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} 1/s_n < \infty$. Show that if $\{x_n\}$ is a sequence of real numbers then the series

$$f(x) = \sum_{n=1}^{\infty} e^{-s_n|x_n - x|}$$

converges a.e. $[m]$.