HW 3

Problems 1–3 are due by Tuesday August 28 in class. The remaining problems, viz., 4–11 are fair game for a quiz in your tutorial on August 29.

General comments: To help the TA mark the HW, please leave margins on the left, and space between each answer. These spaces are often used by markers to make comments (short ones in the margin, and more general ones after your answer).

Notations, conventions, definitions. If X is a topological space, the *support* of a continuous function f on X taking values in $\mathbb{C}$ is the closure of the set $\{f \neq 0\}$. The symbol $C_c(X)$ will denote the space of continuous complex valued functions whose support is compact. If for a continuous function f we write $f \geq \alpha$, or $f \leq \alpha$, or some other inequality, where $\alpha \in \mathbb{R}$, the implicit assumption is that f is *real-valued*, and that the given inequality is true at every point of X.

Locally compact Hausdorff spaces. Recall that a topological space is called *locally compact* if every point in it has a neighbourhood whose closure is compact. We are interested in locally compact Hausdorff spaces. For our purposes the most important theorem concerning such spaces is *Urysohn's Lemma* which states that if V is an open subset of a locally compact Hausdorff space, and K is a compact subset of V, then there exists $f \in C_c(X)$ such that $0 \leq f \leq 1$ on X, $f \equiv 1$ on K, and $f \equiv 0$ on $X \smallsetminus V$.

In the following three exercises, X is a locally compact Hausdorff space, σ a positive measure on $\mathscr{B}(X)$ such that

$$(*) \qquad \int_X f d\sigma < \infty \quad (f \in C_c(X)),$$

and

$$(**) \qquad \sigma(S) = \inf\{\sigma(V) \mid S \subset V,\, V \text{ open}\} \quad (S \in \mathscr{B}(X)).$$

(1) (a) Show that $\sigma(C) < \infty$ for every compact subset C.
 (b) If τ is another measure on $\mathscr{B}(X)$ satisfying $(*)$, $(**)$ and the condition that $\int_X f d\tau = \int_X f d\sigma$ for every $f \in C_c(X)$, then $\tau(C) = \sigma(C)$ for every compact subset C.

(2) Suppose σ satisfies the added hypothesis that

$$\sigma(V) = \sup\Big\{\int_X f d\sigma \mid f \in C_c(X), 0 \leq f \leq 1, \text{support of } f \text{ is in } V\Big\}$$

for every open V. Show that forevery compact C.

$$\sigma(C) = \inf\Big\{\int_X f d\sigma \mid f \in C_c(X),\, 0 \leq f \leq 1,\, f \equiv 1 \,\text{on}\, C\Big\}.$$

(3) Suppose σ satisfies the hypothesis of Problem (2), in addition to $(*)$ and $(**)$. Show that

$$\sigma(V) = \sup\{\sigma(C) \mid C \subset V,\, C \text{ compact}\}$$

for every open set V.

Riemann Integrals on $[0,1]$**.** The integrals in this section are Riemann integrals. In what follows, for a real-valued continuous function f on $I = [0,1]$ set $\|f\|_1 = \int_0^1 |f(t)|dt$ and $\|f\|_2 = \sqrt{\int_0^1 |f(t)|^2 dt}$. You may assume the Cauchy-Schwarz inequality

$$\Big|\int_0^1 f(t)g(t)dt\Big| \leq \|f\|_2 \|g\|_2$$

for continuous real-valued functions f and g on I. You may also use the parallelogram law, namely

$$\|f+g\|_2^2 + \|f-g\|_2^2 = 2\Big(\|f\|_2^2 + \|g\|_2^2\Big).$$

In the exercises below, $\{f_n\}$ is a sequence of continuous functions on I such that $0 \leq f_n \leq 1$, $f_n \to 0$ *pointwise* on I as $n \to \infty$, and $\lim_{n\to\infty} \int_0^1 f_n(t)dt = L$.

You are not allowed to use Lebesgue's Dominated Convergence Theorem for this set of problems since we are dealing with Riemann integrals.

(4) Show that $\|g\|_1 \leq \|g\|_2$ for every continuous g on I.

(5) Let $\{k_n\}$ be a sequence of continuous real-valued functions on I such that $k_n(t) \geq 0$ for all $n \in \mathbb{N}$ and all $t \in I$. Suppose p is a continuous function on I such that $p \geq 0$ on I and $p(t) \leq \sum_{n=1}^\infty k_n(t)$ for every $t \in I$. Show that

$$\int_0^1 p(t)dt \leq \sum_{n=1}^\infty \int_0^1 k_n(t)dt.$$

(Warning: $\sum_{n=1}^\infty k_n(t)$ could diverge for some (even all) $t \in I$.)

(6) For each $n \in \mathbb{N}$, let C_n be the convex hull of $\{f_n, f_{n+1}, \dots\}$, i.e., $g \in C_n$ if and only if $g = \sum_j a_j f_{m_j}$ where $a_j \geq 0$ for all j, $a_j = 0$ for all but a finite number of j, $\sum_j a_j = 1$, and $m_j \geq n$ for all j. Let $d_n = \inf\{\|g\|_2 \mid g \in C_n\}$. Show $\{d_n\}$ is a non-decreasing convergent sequence of real numbers.

(7) Pick $g_n \in C_n$, one for each $n \in \mathbb{N}$. Show that $0 \leq g_n \leq 1$, $g_n \to 0$ pointwise as $n \to \infty$, and $\lim_{n\to\infty} \int_0^1 g_n(t)dt = L$.

(8) For each n, pick $g_n \in C_n$ such that $\|g_n\|_2 \leq d_n + \frac{1}{n}$ (this is possible by definition of d_n). Show using the parallelogram law, that

$$\|g_n - g_m\|_2 \to 0 \text{ as } n, m \to \infty.$$

[*Hint:* Note that $\frac{1}{2}(g_n + g_m) \in C_n$ if $n \leq m$. Why?]

(9) Let $\{g_n\}$ be as in Problem (8). Show that there is a subsequence $\{h_n\}$ of $\{g_n\}$ such that $\sum_{n=1}^\infty \|h_n - h_{n+1}\|_2 < \infty$. For this sequence $\{h_n\}$, show that $h_n = \sum_{m=n}^\infty (h_m - h_{m+1})$.

(10) Let $\{h_n\}$ be as in Problem (9). Show that $\lim_{n\to\infty} \int_0^1 h_n(t)dt = 0$. (Use Problem (5).) Conclude that $L = 0$.

(11) In our assumptions about $\{f_n\}$, drop the assumption that the sequence $\{\int_0^1 f_n(t)dt\}$ converges. Nevertheless show that it must converge, and that the limit is 0. In other words, show that if $\{f_n\}$ is a sequence of continuous functions on I, with $0 \leq f \leq 1$, and such that $f_n \to 0$ *pointwise* on I as $n \to \infty$, then $\lim_{n\to\infty} \int_0^1 f_n(t)dt = 0$. Note that this is trivial once you know DCT. However, it is not trivial to construct the Lebesgue measure, and this set of problems gives a Riemann integral way of doing this. This result is due to Arzela and possibly one of the inspirations for Lebesgue's DCT.