HW 2

General comments: To help the TA mark the HW, please leave margins on the left, and space between each answer. These spaces are often used by markers to make comments (short ones in the margin, and more general ones after your answer).

Definition A map $F\colon \mathbb{R} \to \mathbb{R}$ is called *distribution function* if F is non-decreasing, and right continuous, which means that for each $a \in \mathbb{R}$, $\lim_{x\to a^+} F(x) = F(a)$).

Exercises

(1) Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function. Show that F has at most countable number of points of discontinuity.

(2) Let μ be a positive measure on $\mathbb{R}$ such that $\mu(-\infty, a] < \infty$ for every $a \in \mathbb{R}$ (it is possible that $\mu(\mathbb{R}) = \infty$). Show that $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \mu((-\infty, x]), \qquad x \in \mathbb{R}$$

is a distribution function.

(3) Let F be a distribution function on $\mathbb{R}$, with $\lim_{t\to -\infty} F(t) = 0$. Show that there is a unique measure μ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that

$$F(x) = \mu((-\infty, x]), \qquad x \in \mathbb{R}.$$

Definition: Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. A *normed linear space* V over $\mathbb{K}$ is a vector space over $\mathbb{K}$ together with a function

$$\|\cdot\|\colon V \to [0, \infty)$$

such that

(a) $\|v\| \geq 0$, $v \in V$,
(b) $\|v\| = 0$ if and only if $v = 0$,
(c) $\|\alpha v\| = |\alpha|\|v\|$, $\alpha \in \mathbb{K}$, $v \in V$,
(d) $\|v + w\| \leq \|v\| + \|w\|$, $v, w \in V$.

Let V be a normed linear space over $\mathbb{K}$. As is well-known (if you have not seen this, please check) V is a metric space, the metric given by $d(v, w) = \|v - w\|$. A normed linear space is called a *Banach space* if it is a complete metric space.

(4) In a normed linear space, is the closed ball $\{x \mid \|x - x_0\| \leq r\}$ the closure of the open ball $\{x \mid \|x - x_0\| < r\}$? (Note: This is not always the case for metric spaces.)

(5) Suppose $\{B_n\}$ is a nested sequence of closed balls in a Banach space, i.e.,

$$B_1 \supset B_2 \supset B_3 \supset \cdots \supset B_n \supset B_{n+1} \supset \ldots.$$

Show that $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$.

(6) Show that the normed linear space obtained by putting the norm

$$\|f\| = \int_0^1 |f(t)| dt$$

on the set of continuous functions on $[0, 1]$ is not a Banach space (verify it is a normed linear space).