HW 1

General comments: To help the TA mark the HW, please leave margins on the left, and space between each answer. These spaces are often used by markers to make comments (short ones in the margin, and more general ones after your answer).

Here is a useful notation followed by a useful definition.

For a set X, $\mathscr{P}(X)$ will denote its power set. In other words

$$\mathscr{P}(X) := \{A \mid A \subset X\}.$$

Let X be a non-empty set and $\mathscr{C}$ a subset of $\mathscr{P}(X)$. The smallest σ-algebra in X containing $\mathscr{C}$ is called *the σ-algebra generated by $\mathscr{C}$* and is denoted $\sigma(\mathscr{C})$. In other words, $\sigma(\mathscr{C})$ is characterised by two properties: (i) $\mathscr{C} \subset \sigma(\mathscr{C})$ and (ii) if $\mathcal{A}$ is a σ-algebra containing $\mathscr{C}$, then $\mathcal{A} \supset \sigma(\mathscr{C})$. We will see in Exercise (1) that $\sigma(\mathscr{C})$ exists.

Exercises

(1) Let X be a non-empty set.
  (a) Show that $\mathscr{P}(X)$ and $\{\emptyset, X\}$ are σ-algebras.
  (b) Let $\{\mathcal{A}_\alpha \mid \alpha \in I\}$ be a collection of σ-algebras in X (with I some non-empty index set). Show that
  $$\mathcal{A} := \bigcap_{\alpha \in I} \mathcal{A}_\alpha$$
  is a σ-algebra.
  (c) Let $\mathscr{C}$ be a subset of $\mathscr{P}(X)$. Show that $\sigma(\mathscr{C})$ exists.

Here is another useful definition. Let X be a topological space and τ its collection of open sets. Then $\sigma(\tau)$ is called the *Borel σ-algebra* of X, and is denoted $\mathscr{B}(X)$. In other words $\mathscr{B}(X)$ is the σ-algebra generated by the open sets of X.

(2) Recall that a topological space X is called a T_1-space if for every pair of points x, y in X with $x \neq y$, there exists an open set U such that $x \notin U$, but $y \in U$. Show that in a T_1-space X, every countable subset is in $\mathscr{B}(X)$.

(3) Consider the real line $\mathbb{R}$ with its usual topology. Show that $\mathscr{B}(\mathbb{R})$ is the σ-algebra generated by half-open intervals of the form $(-\infty, a]$, $a \in \mathbb{R}$.

(4) Let $\mathscr{N} \subset \mathscr{P}(\mathbb{R})$ be the set with the property that $N \in \mathscr{N}$ if and only if for every $\varepsilon > 0$ there exists a collection $\{I_n\}_{n\in\mathbb{N}}$ of open intervals such that $\sum_n |I_n| < \varepsilon$ and $N \subset \bigcup_n I_n$. Here for every interval I (whether open, closed, or half-open), the symbol $|I|$ denotes the length of I. Is
$$\mathscr{L}(\mathbb{R}) := \{A \in \mathscr{P}(X) \mid A = B \cup N,\ B \in \mathscr{B}(\mathbb{R}),\ N \in \mathscr{N}\}$$
a σ-algebra? Prove your assertion.