Soln to a HW problem and midterm issues

Suppose μ, ν are measures on $(X, \mathcal{M})$ with $\nu \leq \mu$ and $\mu(X) < \infty$. Let $f \geq 0$ is m'ble and suppose s is a simple function with $0 \leq s \leq f$. Suppose $s = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, $\alpha_i > 0$, E_i's m'ble and pairwise disjoint. Then $\int_X s\, d\nu = \sum_{i=1}^{n} \alpha_i \nu(E_i)$ $\leq \sum_{i=1}^{n} \alpha_i \mu(E_i) = \int_X s\, d\mu \leq \int_X f\, d\mu$. It follows that

$\int_X f\, d\nu \leq \int_X f\, d\mu$. In particular if $\int_X f\, d\mu < \infty$ then $\int_X f\, d\nu < \infty$. By considering f^+ and f^-, if $f \in L^1(\mu)$ we see that $f \in L^1(\nu)$. We therefore have a functional

$$L^1(\mu) \longrightarrow \mathbb{C} \quad \text{given by } f \longmapsto \int_X f\, d\nu.$$

Since $L^2(\mu) \subset L^1(\mu)$, this gives a functional

$$\Phi : L^2(\mu) \longrightarrow \mathbb{C}$$

which is bounded. Indeed

$$|\Phi(f)| = \left|\int_X f\, d\nu\right| \leq \int_X |f|\, d\nu \leq \int_X |f|\, d\mu$$

$$\text{via Cauchy Schwarz} \longrightarrow \leq \|f\|_{2,\mu} \cdot \|1\|_2$$

$$= \|f\|_{2,\mu}\, \mu(X)^{1/2}$$

which means $\|\Phi\| \leq \mu(X)^{1/2}$.

Since $L^2(\mu)$ is a Hilbert space, there exists $g \in L^2(\mu)$, $\|g\|_{2,\mu} = \|\Phi\|$ such that $\Phi(f) = \int_X fg\, d\mu \quad \forall f \in L^2(\mu)$.

In particular, since $\chi_E \in L^2(\mu)$ for $E \in \mathcal{M}$ (for $\mu(X) < \infty$), we have $\Phi(\chi_E) = \int_E g\, d\mu$. On the other hand, by defn, $\Phi(\chi_E) = \nu(E)$. Thus $\nu(E) = \int_E g\, d\mu \quad \forall E \in \mathcal{M}$.

P.T.O.

What can you expect in the exam?

NOT the proof of the Riesz Repⁿ theorem. The HWs and quizzes give a good indication of the kind of problems expected.

Topics covered in tutorials can be used in the solution of problems (e.g. Theorems 1.40, 1.41 on pp. 30-31 of Rudin or the fact that if M is a closed subspace of a Hilbert sp. H, and $x \in H$, $\exists$ a point in M closest to x amongst all points in M).

Definitions of bdd linear maps between normed linear spaces, the norm of a bdd linear transformation, absolute continuity, the concept of $\lambda \perp \mu$ for measures λ and μ, and other Topics which came up in HWs and quizzes are also fair game for the exam. Please make sure you know these too.