Graduate Analysis - I Midterm Exam Solutions

These are brief solutions. Your answers may require more details.

(1) Let $\nu = \{\nu_n\} \in \ell^\infty$. Show that the series $\sum_{n=1}^\infty \nu_n \xi_n$ is absolutely convergent for every $\{\xi_n\} \in \ell^1$. Show that $\{\xi_n\} \mapsto \sum_{n=1}^\infty \nu_n\xi_n$ defines a bounded linear functional $\Lambda_\nu \colon \ell^1 \to \mathbb{C}$ such that $\|\nu\|_\infty = \|\Lambda_\nu\|$, and that every $\Lambda \in (\ell^1)^*$ is equal to Λ_ν for a unique $\nu \in \ell^\infty$. [Hint: Look at $\Lambda(e_i)$, $i \in \mathbb{N}$, where $e_i = \{\chi_{\{i\}}(n)\}$, $i \in \mathbb{N}$.]

Solution. For $\nu = \{\nu_n\} \in \ell^\infty$ and $\{\xi_n\} \in \ell^1$ we have,

$$\sum_n |\nu_n \xi_n| \leq \|\nu\|_\infty \sum_n |\xi_n| = \|\nu\|_\infty \|s\|_1 < \infty.$$

This shows that $\sum_n \nu_n\xi_n$ is absolutely convergent and also that $\|\Lambda_\nu\| \leq \|\nu\|_\infty$. On the other hand, if e_i is the sequence given in the hint, then $\|e_i\|_1 = \sum_n \chi_{\{i\}}(n) = 1$, and $\Lambda_\nu(e_i) = \nu_i$, whence, by definition of $\|\Lambda_\nu\|$ we have:

$$|\nu_i| = \Big|\Lambda_\nu(e_i)\Big| \leq \|\Lambda_\nu\|_\infty \|e_i\|_1 = \|\Lambda_\nu\| \qquad (i \in \mathbb{N}).$$

It follows that $\|\nu\|_\infty \leq \|\Lambda_\nu\|$. Thus $\|\nu\|_\infty = \|\Lambda_\nu\|$. □

(2) Let H be a Hilbert space and $\Lambda \in H^*$. Show there exists a unique element $y_\Lambda \in H$ such that $\Lambda x = \langle x, y_\Lambda\rangle$. Show also that $\|y_\Lambda\| = \|\Lambda\|$. [You may use the fact that any closed subspace of a Hilbert space gives a decomposition of the Hilbert space into the direct sum of the closed subspace and its orthogonal complement. You don't have to prove the existence of such decompositions.]

Solution. If $\Lambda = 0$ there is nothing to prove. So assume $\Lambda \neq 0$. Let $M = \ker\Lambda$. Let $N = M^\perp$. Then M and N are closed subspaces of H, $H = M \oplus N$. Suppose $x \in N$ and $\Lambda(x) = 0$. Then $x \in N \cap M = \{0\}$. Hence $\Lambda|_N$ is injective. Since $\mathbb{C}$ is one dimensional, this forces N to be either 0 or 1-dimensional. Since $\Lambda \neq 0$, therefore $M \neq H$, whence $N \neq 0$. Thus $\Lambda|_N$ is an isomorphism from N to $\mathbb{C}$. It follows there is a unique element $y_0 \in N$ such that $\Lambda(y_0) = 1$. Now if $x \in H$, then $x = m + n$ with $m \in M$, $n \in N$ and this decomposition is unique. Morover, since N is one-dimensional, $n = \alpha y_0$ for a unique $\alpha \in \mathbb{C}$. Thus $\Lambda(x) = \Lambda(m) + \alpha\Lambda(y_0) = 0 + \alpha = \alpha$. If

$$y_\Lambda := \frac{y_0}{\|y_0\|^2},$$

then $\langle x, y_\Lambda\rangle = \langle m, y_\Lambda\rangle + \alpha\langle y_0, y_\Lambda\rangle = \alpha\|y_0\|^2/\|y_0\|^2 = \alpha = \Lambda(x)$. Thus $\langle x, y_\Lambda\rangle = \Lambda(x)$ for every $x \in H$. Uniqueness of y_Λ follows from the fact that if $z \in H$ satisfies $\langle x, z\rangle = 0$ for every $x \in H$ then $z = 0$ (indeed, setting $x = z$, we see that $\|z\|^2 = 0$).

It remains to prove $\|y_\Lambda\| = \|\Lambda\|$. Note that $\|y_\Lambda\| = \|y_0\|^{-1}$. This yields,

$$\begin{aligned}\|\Lambda\|\|y_\Lambda\| &\geq \Big|\Lambda(y_\Lambda)\Big| \\ &= \|y_0\|^{-2} \\ &= \|y_\Lambda\|^2.\end{aligned}$$

It follows that $\|\Lambda\| \geq y_\Lambda$. By Cauchy-Schwarz, we have

$$\Big|\Lambda(x)\Big| = \Big|\langle x, y_\Lambda\rangle\Big| \leq \|x\|\|y_\Lambda\|.$$

By definition of $\|\Lambda\|$, we then have $\|\Lambda\| \leq \|y_\Lambda\|$. □

(3) Let m be the Lebesgue measure on $[0,1]$ and $\{f_n\}$ a sequence of bounded Lebesgue measurable functions on $[0,1]$ satisfying

$$\lim_{n\to\infty}\int_{[0,1]}|f_n|^3 dm = 0.$$

Prove that

$$\lim_{n\to\infty}\int_{[0,1]}\frac{f_n(x)}{\sqrt{x}}dm(x) = 0.$$

Solution. Let $g\colon [0,1]\to[0,\infty]$ be the function given by $g(x) = x^{-1/2}$ for $x\in(0,1]$ and with $g(0)=\infty$. Since $f_n \in L^\infty(m)$ for all $n\in\mathbb{N}$, $\|f_n\|_3 < \infty$ for every natural number n. By Hölder's inequality we therefore have

$$\int_{[0,1]}\frac{f_n(x)}{\sqrt{x}}dm(x) \le \|f_n\|_3\|g\|_{3/2}.$$

(Note $\frac{1}{3}+\frac{2}{3}=1$.) Now since Riemann and Lebesgue integrals coincide for continuous functions on a compact interval and since g is continuous on $[1/k,1]$ for every $k\in\mathbb{N}$, we have

$$\|g\|_{3/2}^{\frac{3}{2}} = \lim_{k\to\infty}\int_{[1/k,1]} g^{\frac{3}{2}}dm = \lim_{k\to\infty}\int_{1/k}^{1} x^{-\frac{3}{4}}dx = 4.$$

Thus

$$\int_{[0,1]}\frac{f_n(x)}{\sqrt{x}}dm(x) \le 4^{\frac{2}{3}}\|f_n\|_3 \to 0 \qquad (\text{as } n\to\infty).$$

□

(4) Let μ be finite (i.e. $\mu(X)<\infty$). Show that for $1\le p\le q$, $L^q(\mu)\subset L^p(\mu)$.

Solution. If $q=\infty$ and $f\in L^q(\mu)$ we have $\int_X |f|^p d\mu \le \|f\|_\infty^p \mu(X) < \infty$, since $\|f\|_\infty<\infty$ and $\mu(X)<\infty$. Thus in this case $f\in L^p(\mu)$. So assume $q<\infty$. Let $f\in L^p(\mu)$ with $p\le q$. Let $A=\{|f|\le 1\}$ and $B=\{|f|>1\}$. Then $|f|^p\chi_A \le \chi_A$ and $|f|^p\chi_B\le |f|^q\chi_B \le |f|^q$. Thus

$$\begin{aligned}
\|f\|_p^p &= \int_X |f|^p\chi_A d\mu + \int_X |f|^p\chi_B d\mu \\
&\le \int_X \chi_A d\mu + \int_X |f|^q d\mu \\
&= \mu(A) + \|f\|_q^q \\
&< \infty.
\end{aligned}$$

□

(5) Let f be a complex measurable function on X such that $\int_E f d\mu = 0$ for every $E \in \mathscr{M}$. Show that $f = 0$ a.e. on X.

Solution. If μ is a finite measure then one can use Theorem 1.40 from Rudin (done in Tutorial 2). However, μ need not be finite. By breaking up f into its real and imaginary parts, it is enough to assume f is real. Let $A = \{f \geq 0\}$. Then $A \in \mathscr{M}$ and $\int_E f^+ d\mu = \int_{A\cap E} f d\mu = 0$ for every $E \in \mathscr{M}$. Similarly $\int_E f^- d\mu = 0$ for every $E \in \mathscr{M}$. Since $f = f^+ - f^-$, it is enough the assume $f \geq 0$. Let $E_n = \{f \geq 1/n\}$ for $n \in \mathbb{N}$. Then

$$0 = \int_{E_n} f d\mu \geq \frac{1}{n}\mu E_n \geq 0,$$

whence $\mu(E_n) = 0$ for every $n \in \mathbb{N}$. This means $\mu(\{f > 0\}) = 0$ since $\{f > 0\} = \bigcup_n E_n$. Since $f \geq 0$, this means $f = 0$ a.e. $[\mu]$. $\square$

(6) Let $\mathscr{F}$ be a σ-algebra on a set Y, and $\phi\colon X \to Y$ a *measurable map*, i.e., $\phi^{-1}(S) \in \mathscr{M}$ for every $S \in \mathscr{F}$. Let $\nu\colon \mathscr{F} \to [0, \infty]$ be the measure given by $\nu(S) = \mu(\phi^{-1}(S))$, for $S \in \mathscr{F}$. Show that $f \in L^1(\nu)$ if and only if $f \circ \phi \in L^1(\mu)$ and that in this case

$$\int_Y f d\nu = \int_X (f \circ \phi) d\mu$$

holds. (You **don't** have to show that ν is a measure.)

Solution. For $S \in \mathscr{F}$ we have $\chi_{\phi^{-1}(S)} = \chi_S \circ \phi$ and hence

$$\int_Y \chi_S d\nu = \nu(S) = \mu\phi^{-1}(S) = \int_X \chi_{\phi^{-1}(S)} d\mu = \int_X (\chi_S \circ \phi) d\mu.$$

The asserted identity

$$(*) \qquad \int_Y f d\nu = \int_X (f \circ \phi) d\mu$$

is therefore true for f a simple measurable non-negative non-negative function (whether f is in $L^1(\nu)$ or not). By MCT, $(*)$ holds for every measurable f such that $f \geq 0$. Since

$$|f| \circ \phi = |f \circ \phi|,$$

it follows from $(*)$ for non-negative functions that $f \in L^1(\nu)$ if and only if $f \circ \phi \in L^1(\mu)$. In this case, since $f^+ \geq 0$ and $f^- \geq 0$, we have

$$\int_Y f^+ d\nu = \int_X (f^+ \circ \phi) d\mu$$

and

$$\int_Y f^- d\nu = \int_X (f^- \circ \phi) d\mu.$$

Now $f^+ \circ \phi = (f \circ \phi)^+$ and $f^- \circ \phi = (f \circ \phi)^-$ and hence $(*)$ holds whenever $f \in L^1(\nu)$ (or, equivalently, whenever $f \circ \phi \in L^1(\mu)$). $\square$

(7) Let $g\colon X \to [0,\infty)$ be measurable and $\nu\colon \mathscr{M} \to [0,\infty]$ the measure $E \mapsto \int_E g d\mu$. Let $A = \{g > 0\}$ and let $f \geq 0$ be a measurable function. Prove that

$$\int_E \frac{f}{g}\, d\nu = \int_{E\cap A} f d\mu$$

for every $E \in \mathscr{M}$.

Solution. We know that $\int_X h d\nu = \int_X hg d\mu$ for all measurable $h \geq 0$ by a theorem done in class. Now $\frac{1}{g} g = \chi_A$, since $\infty \cdot 0 = 0$. Hence

$$\int_E \frac{f}{g}\, d\nu == \int_X \frac{f}{g}\chi_E d\nu = \int_X f\chi_E \frac{1}{g} d\nu = \int_X f\chi_E \frac{1}{g} g d\mu = \int_X f\chi_E\chi_A d\mu = \int_{E\cap A} f d\mu.$$

□

The Lebesgue-Radon-Nikodym decomposition. In the next three problems μ and ν are *finite* measures on $(X,\mathscr{M})$ and $\sigma = \nu + \mu$. The aim of these problems is to prove the essential part of the Radon-Nikodym theorem.

(8) Show that there exists $g \in L^1(\sigma)$, $0 \leq g \leq 1$, such that

$$\mu(E) = \int_E g d\sigma$$

for every $E \in \mathscr{M}$. Show that g is unique as an element of $L^1(\mu)$.

Solution. The uniqueness of g follows from Problem 5. Indeed if $\tilde{g}$ is another measurable function taking values in $[0,1]$ satisfying $\mu(E) = \int_E \tilde{g} d\sigma$ for every $E \in \mathscr{M}$, then

$$\int_E (g - \tilde{g}) d\sigma = 0$$

for every $E \in \mathscr{M}$, whence by Problem 5 $\sigma(\{x \mid g(x) \neq \tilde{g}(x)\}) = 0$. Since $\mu \leq \sigma$, we then get $\mu(\{x \mid g(x) \neq \tilde{g}(x)\}) = 0$. Thus g is unique when regarded as an element of $L^1(\mu)$.

For a measurable complex function f, let $\|f\|_{p,\nu}$, $\|f\|_{p,\mu}$, and $\|f\|_{p,\sigma}$ denote the p-norm of f with respect to the measures ν, μ, and σ respectively (e.g., $\|f\|_{p,\nu} = \left\{\int_X |f|^p d\nu\right\}^{1/p}$). Using simple non-negative functions, it is easy to see, since ν and μ are less than or equal to σ, that $\|f\|_{p,\nu} \leq \|f\|_{p,\sigma}$ and $\|f\|_{p,\mu} \leq \|f\|_{p,\sigma}$, whence if $f \in L^p(\sigma)$, then $f \in L^p(\mu)$. Thus $L^2(\sigma) \subset L^2(\mu) \subset L^1(\mu)$, the last inclusion being true by Problem 4. We therefore have a linear functional

$$\Phi\colon L^2(\sigma) \longrightarrow \mathbb{C}$$

given by $\Phi(f) = \int_X f d\mu$, $f \in L^2(\sigma)$. Now by Hölder (writing $f = fg$, with $g = 1$), we have

$$|\Phi(f)| \leq \|f\|_{2,\mu}\sqrt{\mu(X)} \leq \|f\|_{2,\sigma}\sqrt{\mu(X)}$$

whence $\|\Phi\| \leq \sqrt{\mu(X)} < \infty$, i.e., Φ is bounded. By Problem 2 we have a unique $g \in L^2(\sigma)$ such that

$$\Phi(f) = \int_X fg d\sigma \qquad (f \in L^2(\sigma)).$$

Let $E \in \mathscr{M}$. Taking $f = \chi_E$, which is in $L^2(\sigma)$, for σ is finite, we get

$$\mu(E) = \int_E g d\sigma \qquad (E \in \mathscr{M}).$$

Since $L^2(\sigma) \subset L^2(\mu) \subset L^1(\mu)$ (as we observed above) we see that $g \in L^1(\mu)$.

Finally, since σ is a finite measure, Theorem 1.40 of Rudin (see Tutorial 2 notes of Naageswaran) applies. If $E \in \mathscr{M}$ is such that $\sigma(E) > 0$ then

$$\frac{1}{\sigma(E)} \int_E g d\sigma = \frac{\mu(E)}{\sigma(E)} \in [0,1].$$

It follows (from the quoted result) that $B = \{x \mid g(x) \notin [0,1]\}$ has σ measure zero. One can redefine g by setting $g(x) = 0$ for x in the σ-null set B to get $0 \leq g(x) \leq 1$ for all $x \in X$. □

(9) Let g be as in Problem (8). Let $S = \{g = 0\}$. Let ν_s be the measure on $\mathscr{M}$ given by $\nu_s(E) = \nu(E \cap S)$. Show that $\nu_s \perp \mu$.

Solution. By definition of S, $g\chi_S = 0$, and hence

$$\mu(S) = \int_S g d\sigma = \int_X g\chi_S d\sigma = 0.$$

On the other hand, if $A = X \smallsetminus S$, then $\nu_s(A) = \nu(A \cap S) = \nu(\emptyset) = 0$. Thus $\nu_s \perp \mu$. □

(10) Let g be as in Problem (8). Let $A = \{g > 0\}$ and let ν_a be the measure on $\mathscr{M}$ given by $\nu_a(E) = \nu(E \cap A)$. Find a decreasing function $\phi\colon [0,1] \to [0,\infty]$ such that the following three properties hold:
(a) $\phi = p/q$, where p and q are real polynomials,
(b) $\{\phi = 0\} = \{1\}$,
(c) for $E \in \mathscr{M}$ the following holds:

$$\nu_a(E) = \int_E (\phi \circ g) d\mu.$$

Solution. Take ϕ to be the rational function

$$\phi(x) = \frac{1-x}{x}.$$

It is clear that ϕ satisfies (a) and (b). As for property (c), let $E \in \mathscr{M}$. We have,

$$\begin{aligned}
\int_E (\phi \circ g) d\mu &= \int_E \frac{1-g}{g} d\mu \\
&= \int_{A \cap E} (1-g) d\sigma \qquad \text{(by Problem 7)} \\
&= \sigma(A \cap E) - \int_{A \cap E} g d\sigma \\
&= \sigma(A \cap E) - \mu(A \cap E) \\
&= \nu(A \cap E) \\
&= \nu_a(E).
\end{aligned}$$

□

Remark: This means that $\nu_a \ll \mu$. Note that $\nu = \nu_a + \nu_s$ since A and S are disjoint and their union is X. Thus we have a decomposition of ν into the sum of two measures, one absolutely continuous with respect to μ and the other singular with respect to μ. It is very easy to see that such a decomposition is unique.