Graduate Analysis -I
Semester 1, 2018-19
Midterm Exam

September 27, 2018

Throughout $(X, \mathscr{M}, \mu)$ will denote a measure space. By a measurable function f (unless otherwise stated) we mean a measurable function on $(X, \mathscr{M})$. Similarly a measurable set (unless otherwise stated) means a member of $\mathscr{M}$.

For any set Z, $\mathscr{P}(Z)$ denotes the power set of Z.

If S is a measurable subset of $\mathbb{R}^n$, the Lebesgue measure on S is the restriction of the Lebesgue measure on $\mathbb{R}^n$ to the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^n$ contained in S.

For $1 \leq p \leq \infty$, the symbol ℓ^p will denote the Banach space $L^p(\#)$ where $\#$ is the counting measure on $(\mathbb{N}, \mathscr{P}(\mathbb{N}))$, and as always the norm on ℓ^p is denoted $\|\cdot\|_p$. Note that for $1 \leq p < \infty$, ℓ^p is the set of sequences $s = \{\xi_n\}$ of complex numbers such that $\sum_1^\infty |\xi_n|^p < \infty$ and that for such an s, $\|s\|_p = \{\sum_{n=1}^\infty |\xi_n|^p\}^{\frac{1}{p}}$. The space ℓ^∞ is clearly the space of bounded sequences $s = \{\xi_n\}$ of complex numbers and $\|s\|_\infty = \sup_n |\xi_n|$.

For X and Y normed linear spaces, $B(X, Y)$ will denote the space of bounded linear maps from X to Y. All our normed linear spaces this exam are over $\mathbb{C}$. The space $B(X, \mathbb{C})$, i.e., the space of bounded linear functionals on X, will be denoted X^*. Here $\mathbb{C}$ is regarded as a Banach space in the obvious way. Recall that $B(X, Y)$ is naturally a normed linear space, and it is a Banach space if Y is Banach.

If $h\colon X \to [0, \infty)$ is a measurable function, then $\frac{1}{h}\colon X \to [0, \infty]$ is the map on X which is the usual reciprocal map on $\{h > 0\}$ and takes value ∞ on $\{h = 0\}$. Note that h is measurable. If $f\colon X \to [0, \infty]$ is another measurable map, then $\frac{f}{h}$ is defined to be $f\frac{1}{h}$, with the usual conventions for the multiplication of various combinations of ∞, 0, and finite real numbers.

Each question is worth 10 marks.

(1) Let $\nu = \{\nu_n\} \in \ell^\infty$. Show that the series $\sum_{n=1}^\infty \nu_n \xi_n$ is absolutely convergent for every $\{\xi_n\} \in \ell^1$. Show that $\{\xi_n\} \mapsto \sum_{n=1}^\infty \nu_n \xi_n$ defines a bounded linear functional $\Lambda_\nu\colon \ell^1 \to \mathbb{C}$ such that $\|\nu\|_\infty = \|\Lambda_\nu\|$, and that every $\Lambda \in (\ell^1)^*$ is equal to Λ_ν for a unique $\nu \in \ell^\infty$. [Hint: Look at $\Lambda(e_i)$, $i \in \mathbb{N}$, where $e_i = \{\chi_{\{i\}}(n)\}$, $i \in \mathbb{N}$.]

(2) Let H be a Hilbert space and $\Lambda \in H^*$. Show there exists a unique element $y_\Lambda \in H$ such that $\Lambda x = \langle x, y_\Lambda \rangle$. Show also that $\|y_\Lambda\| = \|\Lambda\|$. [You may use the fact that any closed subspace of a Hilbert space gives a decomposition of the Hilbert space into the direct sum of the closed subspace and its orthogonal complement. You don't have to prove the existence of such decompositions.]

(3) Let m be the Lebesgue measure on $[0, 1]$ and $\{f_n\}$ a sequence of bounded Lebesgue measurable functions on $[0, 1]$ satisfying

$$\lim_{n\to\infty} \int_{[0,1]} |f_n|^3 dm = 0.$$

Prove that

$$\lim_{n\to\infty} \int_{[0,1]} \frac{f_n(x)}{\sqrt{x}} dm(x) = 0.$$

(4) Let μ be finite (i.e. $\mu(X) < \infty$). Show that for $1 \leq p \leq q$, $L^q(\mu) \subset L^p(\mu)$.

(5) Let f be a complex measurable function on X such that $\int_E f d\mu = 0$ for every $E \in \mathscr{M}$. Show that $f = 0$ a.e. on X.

(6) Let $\mathscr{F}$ be a σ-algebra on a set Y, and $\phi\colon X \to Y$ a *measurable map*, i.e., $\phi^{-1}(S) \in \mathscr{M}$ for every $S \in \mathscr{F}$. Let $\nu\colon \mathscr{F} \to [0,\infty]$ be the measure given by $\nu(S) = \mu(\phi^{-1}(S))$, for $S \in \mathscr{F}$. Show that $f \in L^1(\nu)$ if and only if $f \circ \phi \in L^1(\mu)$ and that in this case

$$\int_Y f d\nu = \int_X (f \circ \phi) d\mu$$

holds. (You **don't** have to show that ν is a measure.)

(7) Let $g\colon X \to [0,\infty)$ be measurable and $\nu\colon \mathscr{M} \to [0,\infty]$ the measure $E \mapsto \int_E g d\mu$. Let $A = \{g > 0\}$ and let $f \geq 0$ be a measurable function. Prove that

$$\int_E \frac{f}{g}\, d\nu = \int_{E \cap A} f d\mu$$

for every $E \in \mathscr{M}$.

The Lebesgue-Radon-Nikodym decomposition. In the next three problems μ and ν are *finite* measures on $(X, \mathscr{M})$ and $\sigma = \nu + \mu$. The aim of these problems is to prove the essential part of the Radon-Nikodym theorem.

(8) Show that there exists $g \in L^1(\sigma)$, $0 \leq g \leq 1$, such that

$$\mu(E) = \int_E g d\sigma$$

for every $E \in \mathscr{M}$. Show that g is unique as an element of $L^1(\mu)$.

(9) Let g be as in Problem (8). Let $S = \{g = 0\}$. Let ν_s be the measure on $\mathscr{M}$ given by $\nu_s(E) = \nu(E \cap S)$. Show that $\nu_s \perp \mu$.

(10) Let g be as in Problem (8). Let $A = \{g > 0\}$ and let ν_a be the measure on $\mathscr{M}$ given by $\nu_a(E) = \nu(E \cap A)$. Find a decreasing function $\phi\colon [0,1] \to [0,\infty]$ such that the following three properties hold:
   (a) $\phi = p/q$, where p and q are real polynomials,
   (b) $\{\phi = 0\} = \{1\}$,
   (c) for $E \in \mathscr{M}$ the following holds:

$$\nu_a(E) = \int_E (\phi \circ g) d\mu.$$

Remark: This means that $\nu_a \ll \mu$. Note that $\nu = \nu_a + \nu_s$ since A and S are disjoint and their union is X. Thus we have a decomposition of ν into the sum of two measures, one absolutely continuous with respect to μ and the other singular with respect to μ. It is very easy to see that such a decomposition is unique.