Graduate Analysis - I
Semester 1, 2018-19
Final Exam

(1) (a) (3 marks) Let H be a non-zero Hilbert space and call an orthonormal set $\{u_\alpha \mid \alpha \in A\}$ in H a *complete* orthonormal set if it is a maximal orthonormal set. Give two other equivalent conditions for $\{u_\alpha\}_{\alpha \in A}$ to be complete.

(b) (3 marks) State Tonelli's theorem for **complete** measure spaces.

(c) (4 marks) Let X be a locally compact Hausdorff Space. Define $C_0(X)$ as a normed linear space. State the Riesz Representation Theorem for bounded functionals on $C_0(X)$.

Solution: These are in the notes for the course.

(2) For $1 \leq p \leq \infty$ let $L^p = L^p([0,1], m)$ where m is the Lebesgue measure and let $J\colon L^1 \hookrightarrow (L^\infty)^*$ be the canonical embedding of a normed linear space into its double dual. Let $\lambda\colon C[0,1] \to \mathbb{C}$ be the map $f \mapsto f(1/2)$.

(a) (3 marks) Prove there exists $\Lambda \in (L^\infty)^*$, with $\|\Lambda\| = 1$, such that $\Lambda|_{C[0,1]} = \lambda$.

Solution: Since $|\lambda(f)| = |f(1/2)| \leq \|f\|_\infty$ for $f \in C[0,1]$, therefore $\|\lambda\| \leq 1$. Taking $f = 1$, we see that $\|\lambda\| = 1$. The required $\Lambda \in (L^\infty)^*$ is now obtained by applying the Hahn-Banach theorem.

(b) (7 marks) Show that if Λ is as in (a), then $\Lambda \notin J(L^1)$.

Solution: For $h \in L^\infty$, let Φ_h be symbol we use when we regard h as an element of $(L^1)^*$ under the natural identification of L^∞ with $(L^1)^*$. This means that for $g \in L^1$, $\Phi_h(g) = \int_0^1 hg\,dm$.
Suppose $\Lambda \in J(L^1)$, say $\Lambda = J(g_\Lambda)$. Then $g_\Lambda \in L^1$ and for $h \in L^\infty$, $\Lambda(h) = \Phi_h(g_\Lambda) = \int_0^1 hg_\Lambda\,dm$. In particular we have

$$\lambda(f) = \int_0^1 fg_\Lambda\,dm, \qquad (f \in C[0,1]).$$

On the other hand

$$\lambda(f) = f(1/2) = \int_0^1 f\,d\delta_{1/2}, \qquad (f \in C[0,1]),$$

where $\delta_{1/2}$ is the Dirac measure at $1/2$. The measure represented by $g_\Lambda\,dm$ as well as the measure $\delta_{1/2}$ are regular measures, and hence by the Riesz Representation Theorem, they are equal since both represent λ. This means $\delta_{1/2}$ is absolutely continuous with respect to m, which is a contradiction.

(3) Let X be compact Hausdorff. Show that if $C(X)$ is reflexive then X is finite. (The converse is obvious. Please do not waste your time proving it.)

Solution: Choose $x_0 \in X$. Identify $C(X)^*$ with regular complex measures on X via the Riesz Representation Theorem. Define $\Lambda\colon C(X)^* \to \mathbb{C}$ by the formula

$$\Lambda(\mu) = \mu(\{x_0\})$$

for μ a regular complex measure on X. Since $|\mu(\{x_0\})| \leq |\mu|(X) = \|\mu\|$, therefore Λ is bounded. Now $C(X)$ is reflexive and hence there exists a continuous function f on X such that

$$\Lambda(\mu) = \int_X f\, d\mu \qquad (\mu \in C(X)^*).$$

For $x \in X$, we have

$$\chi_{\{x_0\}}(x) = \delta_x(\{x_0\}) = \Lambda(\delta_x) = \int_X f\, d\delta_x = f(x).$$

Thus $\chi_{\{x_0\}} = f$ and hence $\chi_{\{x_0\}}$ is continuous. Now

$$X \setminus \{x_0\} = \chi_{\{x_0\}}^{-1}(0)$$

and hence $X \setminus \{x_0\}$ is a closed set. Thus $\{x_0\}$ is open, i.e. X is discrete. Since X is compact, this forces X to be finite.

(4) Use the Riesz Representation Theorem for $C_0(X)^*$, for X a locally compact Hausdorff space, to show that there is an isomorphism from ℓ^1 onto c_0^* which preserves norms, and give the isomorphism. Here c_0 is the space of complex sequences $\{s_n\}$ with supremum norm such that $\lim_{n\to\infty} s_n = 0$.

Solution: We can identify c_0 with $C_0(\mathbb{N})$, where $\mathbb{N}$ is given the discrete topology. By the Riesz Representation Theorem c_0^* can be identified isometrically with the space of complex measures on $\mathbb{N}$ (complex measures on $\mathbb{N}$ are obviously regular). The norm of a complex measure μ is its total variation $|\mu|(\mathbb{N})$. Thus, given $\Lambda \in c_0^*$, there exists a unique complex measure μ_Λ on $\mathbb{N}$ such that $\Lambda(s) = \int_{\mathbb{N}} s\, d\mu_\Lambda$ for $s \in c_0$, and in this case $\|\Lambda\| = |\mu_\Lambda|(\mathbb{N})$. Conversely, given a complex measure μ on $\mathbb{N}$, there is a unique $\Lambda \in c_0^*$ such that $\mu = \mu_\Lambda$.

Given $f = \{f_n\} \in \ell^1$, we have the complex measure $\mu = \mu^f$ defined by $d\mu = f\, d\#$, where # is the counting measure on $\mathbb{N}$. Moreover, from a theorem done in class,

$$(*) \qquad \|\mu^f\| = |\mu^f|(\mathbb{N}) = \int_{\mathbb{N}} |f|\, d\# = \sum_{n=1}^{\infty} |f_n| = \|f\|_1.$$

From our earlier discussion, it follows that if

$$\Lambda_f \colon c_0 \longrightarrow \mathbb{C}$$

is the functional

$$\Lambda_f(s) = \int_{\mathbb{N}} s\, d\mu^f = \int_{\mathbb{N}} sf\, d\# = \sum_n s_n f_n \quad (s = \{s_n\} \in c_0),$$

then

$$\|\Lambda_f\| = \|f\|_1.$$

Now every complex measure μ on $\mathbb{N}$ is necessarily absolutely continuous with respect to the counting measure #, for the only subset of $\mathbb{N}$ with zero # measure is the empty set. Thus if μ is a complex measure on $\mathbb{N}$ it must be of the form $d\mu = f\,d\#$ for a unique $f \in L^1(\#) = \ell^1$. In other words $\mu = \mu^f$ for a unique $f \in \ell^1$ and in this case $\|\mu\| = \|f\|_1$.

It follows that $f \mapsto \Lambda_f$ gives us an isometric isomorphism from ℓ^1 to c_0^*.

(5) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution: Let $f(x) = x$ on $[\pi,\, \pi]$. Then $f \in L^2(T)$. Work out $\|f\|_2^2 = 1/(2\pi)\int_{-\pi}^{\pi} x^2 dx$ and the Fourier coefficents $\widehat{f}(n)$, $n \in \mathbb{Z}$, and then use the Parseval identity

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2$$

to get the answer. Details are left to you.

(6) For $n \in \mathbb{N}$, let $\mathscr{B}_n$, $\mathscr{L}_n$ and m_n be the Borel σ-algebra, the Lebesgue σ-algebra, and the Lebesgue measure respectively on $\mathbb{R}^n$. Let $r + s = k$, $r, s \in \mathbb{N}$. Assume $\mathscr{B}_r \times \mathscr{B}_s = \mathscr{B}_k$, that $\mathscr{B}_k = \mathscr{B}_r \times \mathscr{B}_s \subset \mathscr{L}_r \times \mathscr{L}_s$ and that $m_r \times m_s = m_k$ on $\mathscr{L}_r \times \mathscr{L}_s$. Show that $\mathscr{L}_r \times \mathscr{L}_s \subset \mathscr{L}_k$ and that $\mathscr{L}_r \times \mathscr{L}_s \neq \mathscr{L}_k$

Solution: This was done in class using monotone classes.

Fourier Integrals. In the remaining problems $\mathscr{L}$ will be the Lebesgue σ-algebra on $\mathbb{R}$ and m will denote the measure on $\mathscr{L}$ given by

$$m = \frac{1}{\sqrt{2\pi}}(\text{Lebesgue measure}).$$

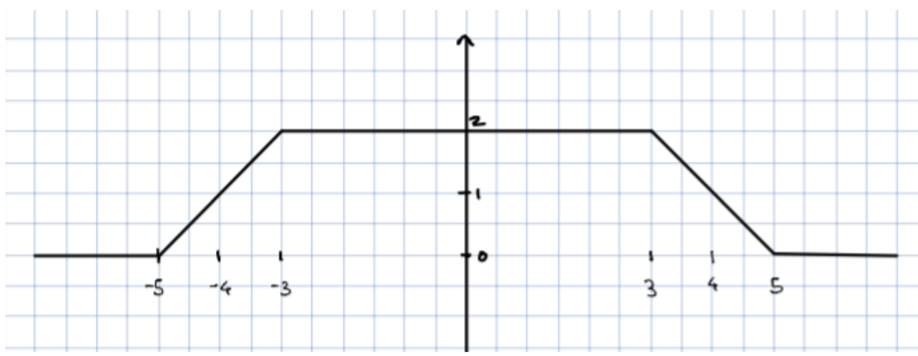
For $1 \le p \le \infty$, L^p will denote $L^p(m)$, and $\|\cdot\|_p$ will denote the norm on L^p. For $f \in L^1$, $\widehat{f}$ denotes the Fourier transform of f.

In what follows, you may use the following easily established fact (please do work it out in your spare time after the exam). For $n \in \mathbb{N}$, let $h_n = \chi_{[-n,n]} * \chi_{[-1,1]}$. Then h_n is continuous and piecewise linear, which is zero in $(-\infty, -n-1] \cup [n+1, \infty)$, is the constant $\sqrt{2/\pi}$ in $[-n+1, n-1]$, and is obvious linear interpolation of these in the $[-n-1, -n+1]$ and $[n-1, n+1]$. In other words, $\|h_n\|_\infty = \sqrt{2/\pi}$ for all $n \in \mathbb{N}$.

For the record:

$$h_n(t) = \begin{cases} 0, & |t| \geq n+1 \\[2ex] \dfrac{n+1+t}{\sqrt{2\pi}}, & -n-1 < t < -n+1 \\[2ex] \sqrt{\dfrac{2}{\pi}}, & 1-n \leq t \leq n-1 \\[2ex] \dfrac{n+1-t}{\sqrt{2\pi}}, & n-1 < t < n+1 \end{cases}$$

Here is the graph of $(\sqrt{2\pi})h_4$:



You don't have to prove any of the above. This is more to get you comfortable with h_n and use it to prove what is asked in the following pages. You will only need to know that $h_n \in C_0(\mathbb{R})$, that $h_n = \chi_{[-n,n]} * \chi_{[-1,1]}$, and that $\|h_n\|_\infty = \sqrt{2/\pi}$.

(7) Prove that
$$\int_{-\infty}^{\infty} \frac{\sin^2(nx)}{x^2}\, dx = n\pi$$
by showing that $\widehat{\chi_{[-n,n]}}(t) = \sqrt{\frac{2}{\pi}}\frac{\sin(nt)}{t}$ and then using Fourier theory. (Other methods will not fetch you marks.)

Solution: Note that $\chi_{[-n,n]} \in L^1 \cap L^2$. Hence $\widehat{\chi_{[-n,n]}}$ exists and is in L^2 and

(∗) $$\|\chi_{[-n,n]}\|_2^2 = \|\widehat{\chi_{[-n,n]}}\|_2^2.$$

We have

$$\sqrt{2\pi}\widehat{\chi_{[-n,n]}}(t) = \int_{-n}^{n} e^{-itx}\,dx = \left[\frac{e^{-itx}}{-it}\right]_{x=-n}^{n}$$
$$= \frac{e^{itn} - e^{-itn}}{it}$$
$$= \frac{2\sin(nt)}{t}$$

It follows that

$$\widehat{\chi_{[-n,n]}}(t) = \sqrt{\frac{2}{\pi}}\frac{\sin(nt)}{t}.$$

The above together with $(*)$ then implies

$$\|\chi_{[-n,n]}\|_2^2 = \frac{1}{\sqrt{2\pi}}\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2(nx)}{x^2}\,dx$$
$$= \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin^2(nx)}{x^2}\,dx$$

On the other hand

$$\|\chi_{[-n,n]}\|_2^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\chi_{[-n,n]}^2(x)\,dx$$
$$= \sqrt{\frac{2}{\pi}}n.$$

Thus

$$\frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin^2(nx)}{x^2}\,dx = \sqrt{\frac{2}{\pi}}n.$$

It follows that

$$\int_{-\infty}^{\infty}\frac{\sin^2(nx)}{x^2}\,dx = n\pi$$

as required.

(8) (a) (3 marks) Let $f_n(x) = \frac{2}{\pi}\frac{\sin(nx)\sin(x)}{x^2}$. Show that $\widehat{f_n} = h_n$ where h_n has been defined two pages earlier.

Solution: Since $h_n = \chi_{[-n,n]} * \chi_{[-1,1]}$ we have

$$\widehat{h_n}(t) = \widehat{\chi_{[-n,n]}}(t)\cdot\widehat{\chi_{[-1,1]}}(t)$$
$$= \frac{2}{\pi}\frac{\sin(nt)\sin(t)}{t^2}$$
$$= f_n(t)$$

Since $\widehat{\chi_{[-n,n]}}$ and $\widehat{\chi_{[-1,1]}}$ are in L^2, their product, namely $\widehat{h_n} = f_n$, is in L^1. Since $\widehat{h_n} \in L^1$, Fourier inversion applies and we have:

$$h_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(t) e^{itx}\, dt = \widehat{g_n}(x)$$

where $g_n(t) = f_n(-t)$. Now $f_n(t) = \frac{2}{\pi}\frac{\sin(nt)\sin(t)}{t^2} = f_n(-t) = g_n(t)$. This proves that $\widehat{f_n} = h_n$.

(b) (7 marks) Show that the Fourier transform $\Phi\colon L^1 \to C_0(\mathbb{R})$, $\Phi(f) = \widehat{f}$, is not an onto map by showing that if f_n is as in part (a), then $\|f_n\|_1 \to \infty$ as $n \to \infty$. Why would this prove that Φ is not onto? (To show $\|f_n\|_1 \to \infty$ as $n \to \infty$, you may use the fact that $\sin(x) \geq x/2$ in $[0,1]$ and the fact that $\int_0^\infty (|\sin(x)|/x)dx = \infty$. You don't have to prove these well known results.)

Solution: Let us first prove that $\|f_n\|_1 \to \infty$ as $n \to \infty$. We have

$$\begin{aligned}
\|f_n\|_1 &= \frac{1}{\pi}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left|\frac{\sin(nx)\sin(x)}{x^2}\right| dx \\
&\geq \frac{1}{\pi}\sqrt{\frac{2}{\pi}} \int_0^1 \left|\frac{\sin(nx)\sin(x)}{x^2}\right| dx \\
&\geq \frac{1}{\pi}\sqrt{\frac{2}{\pi}} \int_0^1 \frac{|\sin(nx)|}{x^2}\frac{x}{2}\, dx \qquad (\text{since } \sin x \geq x/2 \text{ on } [0,1]) \\
&= \frac{1}{\pi}\frac{1}{\sqrt{2\pi}} \int_0^1 \frac{|\sin(nx)|}{x}\, dx \\
&= \frac{1}{\pi}\frac{1}{\sqrt{2\pi}} \int_0^n \frac{|\sin(x)|}{x}\, dx.
\end{aligned}$$

The last quantity $\longrightarrow \infty$ as $n \longrightarrow \infty$.

We have seen in class (via the Fourier Inversion theorem) that if the Fourier transform $\widehat{f}$ of a function f is zero, then $f = 0$ in L^1. Therefore Φ is one-to-one. The norm of $C_0(\mathbb{R})$ is the supremum norm $\|\cdot\|_\infty$. Moreover,

$$|\widehat{f}(t)| \leq 1/(2\pi) \int_{-\infty}^{\infty} |f(x)e^{itx}|\, dx = \|f\|_1$$

for $t \in \mathbb{R}$, i.e.,

$$\|\widehat{f}\|_\infty \leq \|f\|_1.$$

Thus Φ is a continuous linear operator. According to the Open Mapping Theorem, if Φ is onto there exists $\delta > 0$ such that $\|\widehat{f}\|_\infty \geq \delta\|f\|_1$ for every $f \in L^1$. In particular

$$\|h_n\|_\infty \geq \delta\|f_n\|_1$$

for every $n \in \mathbb{N}$. This means $\sqrt{\frac{2}{\pi}} \geq \delta\|f_n\|_1$ for every $n \in \mathbb{N}$, contradicting the fact that $\|f_n\|_1 \to \infty$ as $n \to \infty$.