Graduate Analysis - I
Semester 1, 2018-19
Final Exam

November 29, 2018 Name: ________________________________

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. Number: __________

The exam has eight questions and has 18 pages in all, including this page (9 sheets). Please make sure you have the right number of questions. Question 1 has three parts, Question 2 two parts, and Question 8 has two parts. The remaining questions are single part questions. **Please do not write on this page below this line.**

Question	Maximum marks	Marks
1(a)	3	
1(b)	3	
1(c)	4	
2 (a)	3	
2 (b)	7	
3	10	
4	10	
5	10	
6	10	
7	10	
8(a)	3	
8(b)	7	
Total	80	

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(1) (a) (3 marks) Let H be a non-zero Hilbert space and call an orthonormal set $\{u_\alpha \mid \alpha \in A\}$ in H a *complete* orthonormal set if it is a maximal orthonormal set. Give two other equivalent conditions for $\{u_\alpha\}_{\alpha \in A}$ to be complete.

(b) (3 marks) State Tonelli's theorem for **complete** measure spaces.

(c) (4 marks) Let X be a locally compact Hausdorff Space. Define $C_0(X)$ as a normed linear space. State the Riesz Representation Theorem for bounded functionals on $C_0(X)$.

(2) For $1 \leq p \leq \infty$ let $L^p = L^p([0,\ 1], m)$ where m is the Lebesgue measure and let $J\colon L^1 \hookrightarrow (L^\infty)^*$ be the canonical embedding of a normed linear space into its double dual. Let $\lambda\colon C[0,1] \to \mathbb{C}$ be the map $f \mapsto f(1/2)$.

(a) (3 marks) Prove there exists $\Lambda \in (L^\infty)^*$, with $\|\Lambda\| = 1$, such that $\Lambda|_{C[0,1]} = \lambda$.

(b) (7 marks) Show that if Λ is as in (a), then $\Lambda \notin J(L^1)$.

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(3) Let X be compact Hausdorff. Show that if $C(X)$ is reflexive then X is finite. (The converse is obvious. Please do not waste your time proving it.)

(4) Use the Riesz Representation Theorem for $C_0(X)^*$, for X a locally compact Hausdorff space, to show that there is an isomorphism from ℓ^1 onto c_0^* which preserves norms, and give the isomorphism. Here c_0 is the space of complex sequences $\{s_n\}$ with supremum norm such that $\lim_{n\to\infty} s_n = 0$.

(5) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(6) For $n \in \mathbb{N}$, let $\mathscr{B}_n$, $\mathscr{L}_n$ and m_n be the Borel σ-algebra, the Lebesgue σ-algebra, and the Lebesgue measure respectively on $\mathbb{R}^n$. Let $r + s = k$, $r, s \in \mathbb{N}$. Assume $\mathscr{B}_r \times \mathscr{B}_s = \mathscr{B}_k$, that $\mathscr{B}_k = \mathscr{B}_r \times \mathscr{B}_s \subset \mathscr{L}_r \times \mathscr{L}_s$ and that $m_r \times m_s = m_k$ on $\mathscr{L}_r \times \mathscr{L}_s$. Show that $\mathscr{L}_r \times \mathscr{L}_s \subset \mathscr{L}_k$ and that $\mathscr{L}_r \times \mathscr{L}_s \neq \mathscr{L}_k$

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Fourier Integrals. In the remaining problems $\mathscr{L}$ will be the Lebesgue σ-algebra on $\mathbb{R}$ and m will denote the measure on $\mathscr{L}$ given by

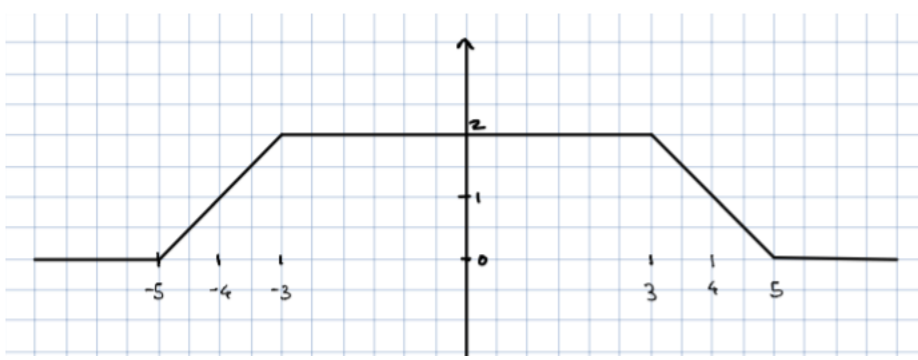
$$m = \frac{1}{\sqrt{2\pi}}(\text{Lebesgue measure}).$$

For $1 \le p \le \infty$, L^p will denote $L^p(m)$, and $\| \cdot \|_p$ will denote the norm on L^p. For $f \in L^1$, $\widehat{f}$ denotes the Fourier transform of f.

In what follows, you may use the following easily established fact (please do work it out in your spare time after the exam). For $n \in \mathbb{N}$, let $h_n = \chi_{[-n,n]} * \chi_{[-1,1]}$. Then h_n is continuous and piecewise linear, which is zero in $(-\infty, -n-1] \cup [n+1, \infty)$, is the constant $\sqrt{2/\pi}$ in $[-n+1, n-1]$, and is obvious linear interpolation of these in the $[-n-1, -n+1]$ and $[n-1, n+1]$. In other words, $\|h_n\|_\infty = \sqrt{2/\pi}$ for all $n \in \mathbb{N}$. For the record:

$$h_n(t) = \begin{cases} 0, & |t| \ge n+1 \\ \dfrac{n+1+t}{\sqrt{2\pi}}, & -n-1 < t < -n+1 \\ \sqrt{\dfrac{2}{\pi}}, & 1-n \le t \le n-1 \\ \dfrac{n+1-t}{\sqrt{2\pi}}, & n-1 < t < n+1 \end{cases}$$

Here is the graph of $(\sqrt{2\pi})h_4$:



You don't have to prove any of the above. This is more to get you comfortable with h_n and use it to prove what is asked in the following pages. You will only need to know that $h_n \in C_0(\mathbb{R})$, that $h_n = \chi_{[-n,n]} * \chi_{[-1,1]}$, and that $\|h_n\|_\infty = \sqrt{2/\pi}$.

(7) Prove that

$$\int_{-\infty}^{\infty} \frac{\sin^2(nx)}{x^2}\,dx = n\pi$$

by showing that $\widehat{\chi_{[-n,n]}}(t) = \sqrt{\frac{2}{\pi}}\frac{\sin(nt)}{t}$ and then using Fourier theory. (Other methods will not fetch you marks.)

(8) (a) (3 marks) Let $f_n(x) = \frac{2}{\pi}\frac{\sin(nx)\sin(x)}{x^2}$. Show that $\widehat{f_n} = h_n$ where h_n has been defined two pages earlier.

(b) (7 marks) Show that the Fourier transform $\Phi\colon L^1 \to C_0(\mathbb{R})$, $\Phi(f) = \widehat{f}$, is not an onto map by showing that if f_n is as in part (a), then $\|f_n\|_1 \to \infty$ as $n \to \infty$. Why would this prove that Φ is not onto? (To show $\|f_n\|_1 \to \infty$ as $n \to \infty$, you may use the fact that $\sin(x) \geq x/2$ in $[0, 1]$ and the fact that $\int_0^\infty (|\sin(x)|/x)dx = \infty$. You don't have to prove these well known results.)

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