ČECH COMPLEX, TENSOR PRODUCT OF COMPLEXES, $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ AND THE KOSZUL COMPLEX

1. Another version of the Čech complex

Here is another version of the Čech complex, which is very obviously quasi-isomorphic to the alternating Čech complex. This is the version given for example in Hartshorne's *Algebraic Geometry*. This will be the default version we'll use in the course, though if we need to distinguish from either the alternating or the usual Čech cohomology we will tag on the label *Hartshorne* to the complex (with apologies to Hartshorne).

Let X be a topological space and $\mathfrak{U} = (U_i)_{i\in I}$ a covering of X, with the index set I having a fixed well-ordering[1]. Write $U_{i_0\ldots i_p}$ for the intersection $U_{i_0\ldots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Define for any *presheaf* $\mathscr{P}$ on X

$$C^p(\mathfrak{U},\, \mathscr{P}) := \prod_{i_0<\cdots<i_p} \mathscr{P}(U_{i_0\ldots i_p}).$$

Define the coboundary $C^p(\mathfrak{U},\, \mathscr{P}) \to C^{p+1}(\mathfrak{U},\, \mathscr{P})$ in the usual way, namely via the standard simplicial formula:

$$(d^p(s))_{i_0\ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k s_{i_o\ldots\hat{k}\ldots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}.$$

Note that this complex is obviously quasi-isomorphic to the *alternating* Čech complex (the total order gives us a way of picking out a component amongst many equivalent entries differing by a sign of a permutation in $C^\bullet_{\mathrm{alt}}(\mathfrak{U},\, \_)$). If we wish to distinguish the Hartshorne version from other versions, we will use the symbol $C^\bullet_H(\mathfrak{U},\, \mathscr{P})$. Otherwise, we will keep subscripts and superscripts to a minimum, and use $C^\bullet(\mathfrak{U},\, \mathscr{P})$ for the just defined complex.

This version is very useful in computing the cohomology of line bundles on projective space.

2. The tensor product of two complexes and $\mathrm{Hom}^\bullet(A^\bullet,\, B^\bullet)$

2.1. **General nonsense.** Given two categories $\mathscr{A}$ and $\mathscr{B}$, the product category makes sense in an obvious way: the objects are pairs $(A,\, B)$, with A and object of $\mathscr{A}$ and B an object $\mathscr{B}$. Likewise morphisms $(A_1,\, B_1) \to (A_2,\, B_2)$ are pairs $(f,\, g)$, with $f\colon A_1 \to A_2$ and $g\colon B_1 \to B_2$ maps in $\mathscr{A}$ and $\mathscr{B}$ respectively. Definitions of composites etc. are left to you.

Now suppose $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are abelian categories.

A functor $T\colon \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ is a *biadditive covariant functor* if for fixed $A \in \mathscr{A}$, $B \in \mathscr{B}$, the functors $T(A, \bullet)\colon \mathscr{B} \to \mathscr{C}$ and $T(-,\, B)\colon \mathscr{A} \to \mathscr{C}$ are additive covariant functors. An example of this, with A a commutative ring, is the tensor product functor $- \otimes_A \bullet\colon \mathrm{Mod}_A \times \mathrm{Mod}_A \to \mathrm{Mod}_A$.

[1]By the Axiom of Choice, there is at least one well-ordering on I.

Similarly $T\colon \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ is a *biadditive functor, contravariant in the first argument and covariant in the second argument*, if $T(-,\,B)$ is contravariant additive, and $T(A,\,\bullet)$ is covariant additive. An example of this is

$$\mathrm{Hom}_{\mathscr{A}}(-,\,\bullet)\colon \mathscr{A} \times \mathscr{A} \to \mathcal{A}b$$

where $\mathcal{A}b$ is the category of abelian groups. Sometimes it is easier to write such a functor as $T\colon \mathscr{A}^{\circ} \times \mathscr{B} \to \mathscr{C}$, to indicate the contravariance in the first argument (recall, $\mathscr{A}^{\circ}$ is the opposite category of $\mathscr{A}$).

You can work out the definitions for the other two possibilities for additive bifunctors by yourself.

For such functors there are standard conventions for turning $T(A^\bullet,\,B^\bullet)$ into a complex. In the first (biadditive covariant) case we have commutative diagrams (one for each pair of integers $(p,\,q)$):

$$\begin{tikzcd}
T(A^p, B^{q+1}) \arrow[r, "\text{via}\,\partial_{A^\bullet}"] & T(A^{p+1},\, B^{q+1}) \\
T(A^p,\, B^q) \arrow[u, "\text{via}\,\partial_{B^\bullet}"'] \arrow[r, "\text{via}\,\partial_{A^\bullet}"'] & T(A^{p+1},\, B^q) \arrow[u, "\text{via}\,\partial_{B^\bullet}"]
\end{tikzcd}$$

This evidently gives a double complex (provided $\mathscr{C}$ has countable direct sums[2] or some such thing). The total complex is denoted $T^\bullet(A^\bullet,\,B^\bullet)$. In other words

$$(\dagger) \qquad T^\bullet(A^\bullet,\,B^\bullet) := \mathrm{Tot}^\bullet T(A^\bullet,\,B^\bullet).$$

There is one exception to this notation. The total complex of the tensor product of complexes (in categories where $\otimes$ exist) should be denoted $A^\bullet \overset{\bullet}{\otimes} B^\bullet$ by this convention, but instead is denoted $A^\bullet \otimes B^\bullet$.

Similarly in the second case (biadditive, contravariant in the first, covariant in the second) we associate a double complex (assuming $\mathscr{C}$ has enough properties, e.g. countable direct sums etc)

$$T(A^\bullet,\,B^\bullet) = (T(A^{-q},\,B^p),\,T(1,\partial^p_{B^\bullet}),\,(-1)^{q+1}T(\partial^{-q-1}_{A^\bullet},\,1))_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}.$$

Note that for each $(p,\,q)$ we have a commutative diagram:

$$\begin{tikzcd}
T(A^{-q-1}, B^p) \arrow[r, "\text{via}\,\partial_{B^\bullet}"] & T(A^{-q-1},\, B^p) \\
T(A^{-q},\, B^p) \arrow[u, "(-1)^{q+1}T(\partial^{-q-1}_{A^\bullet},\,1)"] \arrow[r, "\text{via}\,\partial_{B^\bullet}"'] & T(A^{-q},\, B^p) \arrow[u, "(-1)^{q+1}T(\partial^{-q-1}_{A^\bullet},\,1)"']
\end{tikzcd}$$

The total complex of this is denoted $T^\bullet(A^\bullet,\,B^\bullet)$. Thus, we once again have a formula:

$$(*) \qquad T^\bullet(A^\bullet,\,B^\bullet) := \mathrm{Tot}^\bullet T(A^\bullet,\,B^\bullet).$$

To summarize notation (for either case; bi-covariant or mixed):

- Double complex $= T(A^\bullet,\,B^\bullet)$.
- Total complex $= T^\bullet(A^\bullet,\,B^\bullet)$.

[2] If $\mathscr{C}$ has countabe direct products, then we can extend the definition of a double complex and its total complex in an obvious way and this is done for the $\mathrm{Hom}(-,\,\bullet)$ biaddtive functor, since $\mathcal{A}b$ has arbitrary products and sums.

2.2. **Tensor product of complexes.** Suppose A is a commutative ring (or more generally a sheaf of rings over a topological space). Then for A-modules M and N we have the biadditive functor $T(M,\, N) = M \otimes_A N$. If $R^\bullet$ and $S^\bullet$ are complexes, then

$$R^\bullet \otimes_A S^\bullet := T^\bullet(R^\bullet,\, S^\bullet)$$

where we are using formula (†) for the right-side. In practical terms,

$$[R^\bullet \otimes_A S^\bullet]^n = \bigoplus_{p+q=n} R^p \otimes_A S^q$$

and the coboundary is:

$$\partial^{p+q}(r^p \otimes s^q) = \partial^p(r^p) \otimes s^q + (-1)^p r^p \otimes \partial^q(s^q).$$

We point out again that consistency demands that we write $R^\bullet \overset{\bullet}{\otimes}_A S^\bullet$ for this complex, but thankfully nobody submits to such notational tyranny.

2.3. **The complex $\mathbf{Hom}^\bullet(A^\bullet,\, B^\bullet)$.** Now suppose $\mathscr{A}$ is an abelian category, and $A^\bullet$, $B^\bullet$ are complexes. Then applying our general principles we have a complex

$$\mathrm{Hom}^\bullet_{\mathscr{A}}(A^\bullet,\, B^\bullet) = T^\bullet(A^\bullet,\, B^\bullet)$$

where of course $T^\bullet$ is given by $(*)$ in the previous subsection with $T(M,\, N) = \mathrm{Hom}_{\mathscr{A}}(M,\, N)$.[3] But this time we make full use of the fact that arbitrary direct products exist for abelian groups. So here is $\mathrm{Hom}^\bullet_{\mathscr{A}}(A^\bullet,\, B^\bullet)$ explictly (and to lighten notation, we drop $\mathscr{A}$ from the subscript for Hom):

$$\mathrm{Hom}^n(A^\bullet,\, B^\bullet) = \prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j,\, B^{j+n}).$$

The coboundary $d^n\colon \mathrm{Hom}^n(A^\bullet,\, B^\bullet) \to \mathrm{Hom}^{n+1}(A^\bullet,\, B^\bullet)$ takes $f = (f^j)_{j\in\mathbb{Z}}$ in $\prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j,\, B^{j+n}) = \mathrm{Hom}^n(A^\bullet,\, B^\bullet)$ to $d^n(f) \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j,\, B^{j+n+1}) = \mathrm{Hom}^{n+1}(A^\bullet,\, B^\bullet)$ and the formula is:

$$d^n(f) = (\partial^{n+j}_{B^\bullet} \circ f^j + (-1)^{n+1} f^{j+1} \circ \partial^j_{A^\bullet})_{j\in\mathbb{Z}}.$$

This formula is standard (see for example B. Iversen's "Cohomology of Sheaves" from which much of today's notes were made or Joseph Lipman and Mitsuyasu Hashimoto's rather advanced "Foundations of Grothendieck Duality for Diagrams of Schemes"). Unfortunately a classic in the subject—Robin Hartshorne's "Residues and Duality"—has a different convention and it differs from the above by a factor of $(-1)^{n+1}$.

3. Koszul Complexes

There are two related Koszul complexes. The *homology Kozsul complex* and the *cohomology Koszul complex*. Let A be a commutative ring and t an element of A and M an A-module. The cohomology Koszul complex of M with respect to t is the complex

$$K^\bullet(t, M): \quad 0 \to M \xrightarrow{t} M \to 0$$

with the left M in the 0^{th} spot and whence the right M in the first spot. Note that

$$K^\bullet(t,\, M) = K^\bullet(t,\, A) \otimes_A M.$$

[3] Note our sudden rediscovery of ideological/notational purity. We now put decorations around Hom, something we didn't for $\otimes$. Think about the reason for this.

If $\mathbf{t} = (t_1, \dots, t_n)$ is a sequence of elements in A, then the *Koszul cohomology complex of M with respect to* $\mathbf{t}$ is

$$K^\bullet(\mathbf{t},\, M) := K^\bullet(t_1,\, A) \otimes_A \dots \otimes_A K^\bullet(t_n,\, A) \otimes_A M.$$

This complex lives in degrees $0,\, 1, \dots, n$.[4]

The homology Koszul complexes are identical expect for the grading involved. Thought of as a co-chain complex it lives in degrees $-n, -n+1, \dots, 0$ (and hence as a chain complex, i.e. as a homology complex, it lives in degrees $0, \dots, n$). Recall that one can turn a cohomology complex $C^\bullet$ into a homology complex $C_\bullet$ by setting $C_i = C^{-i}$. Then the Koszul homology complex of M with respect to $\mathbf{t}$ is obtained by shifting $K^\bullet(\mathbf{t},\, M)$ n-places to the left (no change of sign) and then regarding it as a homology complex. As a chain complex (=homology complex) it is invoked by the symbol $K_\bullet(\mathbf{t},\, M)$. It is easy to see that

$$K_\bullet(\mathbf{t},\, M) = K_\bullet(t_1,\, A) \otimes_A \dots \otimes_A K_\bullet(t_n,\, A) \otimes_A M.$$

An interesting fact is that $K^\bullet(\mathbf{t},\, M) = \mathrm{Hom}_A(K_\bullet(\mathbf{t},\, A),\, M)$ where the complex on the right is the naive complex (without the fancy sign conventions of the previous sections).

These complexes are intimately related to Čech complexes as your Homework will show.

[4]This is an n-fold tensor product of complexes, and rightly should be arranged in an n-dimensional grid. I tried but the two dimensional page just didn't have enough space to squeeze in such a grid, a problem many orders more complicated than the one Fermat faced with his margin. By the way, there is an associative law for tensor product of complexes, and in any case, it is enough to think of $A_1^\bullet \otimes \dots \otimes A_n^\bullet$ as $((\dots((A_1^\bullet \otimes A_2^\bullet) \otimes A_3^\bullet) \dots) \otimes A_{n-1}^\bullet) \otimes A_n^\bullet$.