Aug 30, ~~2010~~ and Sep 3, 2010

Defn (Additive functor): Let $\mathcal{C}$ and $\mathcal{D}$ be ~~additive~~ abelian cats. A functor $T: \mathcal{C} \to \mathcal{D}$ is said to be an additive functor if for each pair of objects A, B in $\mathcal{C}$, the canonical map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(TA, TB)$$

is a group homomorphism.

From now on, working in a fixed abelian cat.

Criterion for splitting an object: Let B be an object. Suppose $f: A \to B$, $g: B \to C$ are maps such that there are maps $\sigma: C \to B$, $\tau: B \to A$ with ~~$g\sigma = 1_C$ and~~

(a) $g\sigma = 1_C$

(b) $\tau f = 1_A$

(c) $f\tau + \sigma g = 1_B$.

Then the maps

$$\varphi = (f \quad \sigma) : A \oplus C \longrightarrow B$$

and

$$\theta = \begin{pmatrix} \tau \\ g \end{pmatrix} : B \longrightarrow A \oplus C$$

are isomorphisms, ~~with~~ in fact inverses of one another.

Proof: From (a) σ is a mono, g an epi. From (b) f is a mono and τ an epi.

Now $g = g \circ 1_B = g(f\tau + \sigma g) = gf\tau + g\sigma g = gf\tau + g$ (since $g\sigma = 1$)

Cancelling g, we get $gf\tau = 0$.

Now τ is an epi, whence

$$gf = 0.$$

Similarly $\sigma = 1_B \circ \sigma = f\tau\sigma + \sigma g \sigma = f\tau\sigma + \sigma$

$\Rightarrow f\tau\sigma = 0 \Rightarrow \tau\sigma = 0$ since is mono.

Thus
$$gf = 0 \quad \text{and} \quad \tau\sigma = 0.$$
Now
$$\varphi\theta = (f\ \sigma)\begin{pmatrix}\tau \\ g\end{pmatrix} = \cancel{fg}\ f\tau + \sigma g = 1_B$$
$$\theta\varphi = \begin{pmatrix}\tau \\ g\end{pmatrix}(f\ \sigma) = \begin{pmatrix}\tau f & \tau\sigma \\ gf & g\sigma\end{pmatrix} = \begin{pmatrix}1_A & 0 \\ 0 & 1_C\end{pmatrix} = 1_{A\oplus C}$$

q.e.d.

<u>Definition</u>: A short exact sequence ~~in an abelian~~
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
is said to be a <u>split exact sequence</u> if there exists an isomorphism
$$\theta : B \xrightarrow{\sim} A\oplus C$$
s.t. the diagram
$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \| & & \theta\downarrow\wr & & \| & & \\ 0 & \to & A & \xrightarrow[i_A]{} & A\oplus C & \xrightarrow[p_C]{} & C & \to & 0 \end{array}$$
commutes, where $i_A : A \to A\oplus C$ is the canonical ~~monomorphism~~ and $p_C : A\oplus C \to C$ the canonical projection. (Note the bottom row is already exact).

<u>Remark</u>: According to our criterion, if $A \xrightarrow{f} B \xrightarrow{g} C$ are a pair of maps (this is not a complex a-priori) with $\sigma : C \to B$ and $\tau : B \to A$ as in the criterion, then
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
is a split exact sequence.

Proposition : Let

$$(*) \qquad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact seq. TFAE

(a) The map $g : B \to C$ has a section $\sigma : C \to B$ (i.e. $g\sigma = 1_C$)

(b) There exists $\tau : B \to A$ s.t. f is a section of τ.

(c) The exact sequence $(*)$ is split.

Proof :

Suppose (a) is true. Let $\pi : B \to B$ be the map

$$\pi = 1_B - \sigma g.$$

Since $g\sigma = 1_C$ and $gf = 0$, one checks that

$$\pi^2 = \pi \quad \text{and } \pi f = f.$$

Claim 1 : $\ker(B \xrightarrow{\pi} B) = \sigma$.

Pf : Let $b : T \to B$ be s.t. $\pi b = 0$. We have to show $\exists!\ c : T \to C$ s.t. $\sigma c = b$. Since σ is a mono (all sections are monos!), uniqueness is clear. As for the existence, set $c = gb$. Then $\sigma c = \sigma g b = (1_B - \pi) b$ $= b - \pi b = b$, since $\pi b = 0$. q.e.d. for claim 1.

Claim 2 : $\operatorname{coker}(B \xrightarrow{\pi} B) = g$.

Pf : Let $t : B \to T$ be a map s.t. $t\pi = 0$. We've to show $\exists!\ t^* : C \to T$ s.t. $t = t^* g$. Consider the series of implications of the true statement $t\pi = 0$.

$t\pi = 0 \Rightarrow t(1_B - \sigma g) = 0 \Rightarrow t - t\sigma g = 0 \Rightarrow tf - t\sigma g f = 0 \Rightarrow tf = 0$ (for $gf = 0$).

Thus $tf = 0$. Use $g = \operatorname{coker}(A \xrightarrow{f} B)$ to conclude $\exists!\ t^* : C \to T$ as required. q.e.d. for claim 2.

~~Clam~~

Claim 3 : $\operatorname{im}(\pi) = f$.

Pf : $\operatorname{im} \pi = \ker(\operatorname{coker} \pi) = \ker(g) = f$ q.e.d. for claim 3.

Since we are in an abelian category, $\mathrm{im}(\pi) = \mathrm{coim}(\pi)$, thus $\exists\ \tau : B \to A$ such that the canonical factorization of $B \to \mathrm{coim}(\pi) \xrightarrow{\sim} \mathrm{im}(\pi) \to A$ gives us a comm. diag

$$\begin{tikzcd} B \arrow[r, "\pi"] \arrow[d, "\tau"'] & B \\ A \arrow[r, equal] & A \arrow[u, "f"'] \end{tikzcd} \qquad (*)$$

Now $\tau = \mathrm{coim}(\pi)$ and $\sigma = \ker(\pi)$ imply that

$$\tau\sigma = 0 \qquad (1)$$

We pointed out earlier that $\pi^2 = \pi$ & $\pi f = f$. Thus

$$f(\tau f) = (f\tau) f = \pi f = f = f \circ 1_A. \qquad (f\tau = \pi \text{ by } (*))$$

Since f is a mono, this gives

$$\tau f = 1_A \qquad (2)$$

Finally, $\quad f\tau = \pi := 1_B - \sigma g \qquad (f\tau = \pi \text{ by } (*))$

$$\Rightarrow \quad f\tau + \sigma g = 1_B \qquad (3).$$

From (1), (2) and (3), and our criterion for splitting, we have shown (a) $\Rightarrow$ (c). (The existence of τ also gives (a) $\Rightarrow$ (b)).

By duality (b) $\Rightarrow$ (a). (since (a) $\Rightarrow$ (b)).

Finally, very trivially (a) and (b) follow from (c).

We will be working in a fixed abelian cat.

Lemma on direct summand of exact sequences: Suppose $A_1, A_2, B_1, B_2, C_1, C_2$ are objects and we have maps $\alpha_i : A_i \to B_i$, $\beta_i : B_i \to C_i$, $i = 1, 2$. Then the sequence

$$0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}} B_1 \oplus B_2 \xrightarrow{\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}} C_1 \oplus C_2 \longrightarrow 0$$

is exact if and only if the sequence

$$0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$$

are exact for $i = 1, 2$.

Proof: Obvious.

Injective and Projective objects:

An object E is said to be *injective* if given a diagram with solid arrows

$$\begin{tikzcd} & & E \\ 0 \arrow[r] & M \arrow[r] \arrow[ur, "\varphi"] & N \arrow[u, dashed] \end{tikzcd} \quad \text{(exact)}$$

the dotted arrow can be filled to make it commute.

The dual notion is that of a *projective* object. An object P is projective if given a diagram of solid arrows (as below), the dotted arrow can be filled to make it commute

$$\begin{tikzcd} P \arrow[d, dashed] \arrow[dr, "\varphi"] & & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \quad \text{(exact).}$$

Definitions: A covariant functor (resp. contravariant functor) $F: \mathcal{A} \to \mathcal{B}$ (resp. $G: \mathcal{A} \to \mathcal{B}$) between abelian categories $\mathcal{A}$ and $\mathcal{B}$ is said to be left exact if every exact sequence of the form

$$0 \to A \to B \to C$$

(resp. $A \to B \to C \to 0$) transforms to an exact sequence

$$0 \to FA \to FB \to FC$$

(resp. $0 \to GC \to GB \to GA$)

F (resp. G) ~~as~~ is right exact if exact sequences of the form

$$A \to B \to C \to 0$$

(resp. $0 \to A \to B \to C$) transform to exact sequences

$$FA \to FB \to FC \to 0$$

(resp. $GC \to GB \to GA \to 0$).

Examples: Fix ~~..~~ an object X. Then
(a) $\mathrm{Hom}(X, -)$ is left exact (covariant)
(b) $\mathrm{Hom}(-, X)$ is also left exact (contravariant).

Example: Let A be a comm. ring with 1. Fix an A module M. Then $(-) \otimes_A M$ is right exact.

Defn: A functor is exact if it is both left and right exact.

Rmk. If F is exact & $C^\bullet$ is an exact complex, then ~~two~~ easy to see that $FC^\bullet$ is exact. (Apply F to $0 \to Z^n(C^\bullet) \to C^n \to Z^{n+1}(C^\bullet) \to 0$ and "splice" together the resulting sequences)

<u>Proposition</u> : Let X be an object.

(a) X is injective if and only if $\mathrm{Hom}(-, X)$ is exact.

(b) X is projective if and only if $\mathrm{Hom}(X, -)$ is exact.

<u>Proof</u>: These are just restatements of the definitions (only have to check right exactness, and this is a restatement of the defn in ~~either~~ each case).

<u>Proposition</u>: (a) Direct summands of injective objects are injective, i.e. if $E = E_1 \oplus E_2$ and E is injective, then E_1 and E_2 are injective.

(b) Direct summands of projective objects are projective.

<u>Proof</u>

(a) Let $E = E_1 \oplus E_2$ be an injective object. Then

$$\mathrm{Hom}(-, E) = \mathrm{Hom}(-, E_1) \oplus \mathrm{Hom}(-, E_2)$$

Apply the Lemma on direct summands of exact sequences given earlier.

(b) Let $P = P_1 \oplus P_2$ be a projective object. $\mathrm{Hom}(P, -) = \mathrm{Hom}(P_1, -) \oplus \mathrm{Hom}(P_2, -)$.

q.e.d.

<u>Proposition</u> : Let A be a comm. ring with 1. Every free module is projective.

<u>Proof</u>: Let F be free. Then $F \cong \bigoplus_{i \in I} A$ for some index set I.

If ~~[illegible]~~ M is an A-module, then clearly

$$\mathrm{Hom}_A(F, M) \cong \prod_{i \in I} \mathrm{Hom}(A, M) \simeq \prod_{i \in I} M.$$

It follows that $\mathrm{Hom}_A(F, -)$ is exact.

~~Proposition: Let A be a comm. ring with 1. An A-module P is projective if and only if it is the direct summand~~

<u>Proposition</u>: Let E be an injective object and P a projective object.

(a) If $E \to X$ is a monomorphism, then E is a direct summand of X.

(b) If $X \to P$ is an epimorphism then P is a direct summand of X.

<u>Proof</u>: Enough, by duality, to prove (a). Consider the problem of solving for the dotted arrow (to make the resulting diag commute).

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & E \arrow[u, equal] \arrow[r] & X \arrow[ul, dotted] \end{tikzcd} \quad \text{(exact)}.$$

Since E is injective, $\exists$ a soln $\varphi : X \to E$.

Now consider the exact seq.

$$0 \to E \to X \to C \to 0 \qquad (C = \mathrm{coker}\,(E \to X))$$

Since we have a map $\varphi : X \to E$ s.t. $E \to X$ is a section of φ, we have a splitting by one of our propositions. q.e.d.

<u>Proposition</u>: Let A be a comm. ring with 1, An A-module P is projective if and only if it is the direct summand of a free module.

<u>Proof</u>: If F is free and $F = P \oplus Q$, ~~we've seen that P (and Q) is projective~~. Since F is projective, and P is a direct ~~Conversely,~~ summand, our previous results give that

P is projective.

Conversely suppose P is projective. We can always find a surjective map $\pi : F \to P$ with F free (e.g. if $\{p_\alpha\}_{\alpha \in I}$ are generators of P, let $F = \bigoplus_{\alpha \in I} R$ and $\pi : F \to P$ the map which sends $(\delta_{\alpha\beta})_{\beta \in I}$ to p_α, $\alpha \in I$. ($\delta_{\alpha\beta}$ = Kronecker delta)). Since P is projective, it must split off from F as a direct summand.

q.e.d.

Injective Modules over Commutative Rings:

Let A be a commutative ring (always with 1), and Mod_A the category of unital A-module (i.e., for $M \in Mod_A$, and $m \in M$, $1 \cdot m = m$).

We will show that Mod_A has "enough injectives" i.e., every $M \in Mod_A$ can be embedded into an injective object of Mod_A. (Injective objects of Mod_A are called injective modules)

Mod_A certainly has "enough projectives" - every module is the quotient of a free module, and free modules are projective, so every module is the quotient of a projective module. (Projective module := Projective object of Mod_A).

First we need:

Theorem (BAER'S CRITERION): An A-module E is injective if and only if every A-map $\varphi : I \to E$ from an ideal $I \subset A$, can be extended to an A-map $\tilde{\varphi} : A \to E$

$$\begin{tikzcd} & & E \\ 0 \arrow[r] & I \arrow[ur, "\varphi"] \arrow[r] & A \arrow[u, dashed, "\tilde{\varphi}"'] \end{tikzcd} \quad \text{(exact)} \quad \left.\begin{array}{l} \text{Problem has soln } \forall\, I \overset{\text{ideal}}{\subset} A \\ \quad \Leftrightarrow \ E \text{ injective.} \end{array}\right.$$

Only need to prove one-way. The other way is obvious.

Proof: (Needs axiom of choice).

Suppose E satisfies Baer's criterion. Let M be a submodule of an A-module N. Suppose

$$\psi : M \longrightarrow E$$

is an A-map. We have to find an extn ~~$\tilde{\psi}: N \to E$~~ ~~of ψ~~ $\tilde{\psi}: N \to E$ of ψ.

Let $M' \subset N$ be ~~the~~ a maximal submodule of N to which ψ can be extended. Such an M' together with an extension $\psi': M' \to E$ of $\psi: M \to N$ exists by Zorn's Lemma (for those unfamiliar with the argument, here's how you set it up: Consider pairs (S, θ), where $S \subset N$ is a submodule ~~and~~ containing M, i.e. ~~$S \supset M$~~ $M \subset S \subset N$, and ~~$\theta|_M$~~ $\theta : S \to E$ is a map s.t. $\theta|_M = \psi$. Decree $(S, \theta) < (S', \theta')$ if $S \subset S'$, and $\theta'|_S = \theta$. Zorn applies to this situation.) So have (M', ψ') max'l amongst extensions of ψ.

If $M' = N$ we are done. If not let $n \in N \setminus M'$. Let

$$I = \{ a \in A \mid an \in M' \}.$$

Then I is an ideal of A (note $1 \notin I$) and we have a map

$$\varphi : I \longrightarrow E$$

defined by

$$\varphi(a) = \psi'(an) \quad , \quad a \in I.$$

~~By Bae~~ Since E satisfies Baer, $\exists$

$$\tilde{\varphi} : A \longrightarrow E$$

s.t. $\tilde{\varphi}|_I = \varphi$. Now define ~~$M'' = M + an$~~

$M'' := M' + A\cdot n$, and $\Psi'' : M'' \longrightarrow E$ by

$$\Psi''(m' + a\cdot n) = \Psi'(m') + \tilde{\varphi}(a), \quad m' \in M',\ a \in A.$$

One checks easily that Ψ'' is well-defined. Indeed, if

$$m_1' + a_1 n = m_2' + a_2 n \quad \text{then} \quad m_1' - m_2' = (a_2 - a_1) n,$$

whence $a_1 - a_2 \in I$. It follows that

$$\begin{aligned}
\tilde{\varphi}(a_1) - \tilde{\varphi}(a_2) &= \tilde{\varphi}(a_1 - a_2) \\
&= \varphi(a_1 - a_2) \qquad (\text{since } \varphi = \tilde{\varphi}|_I) \\
&= \Psi'((a_1 - a_2)\cdot n) \\
&= \Psi'(m_2' - m_1') \\
&= \Psi'(m_2') - \Psi'(m_1')
\end{aligned}$$

Thus

$$\Psi'(m_1') + \tilde{\varphi}(a_1) = \Psi'(m_2') + \tilde{\varphi}(a_2),$$

i.e., $\Psi'' : M'' \to E$ is well-defined, and we have ~~contr~~ violated the maximality of (M', Ψ').

Thus $M' = N$.

q.e.d.

<u>Divisible modules over P.I.Ds</u> : Let A be a P.I.D..

A module M over A is said to be <u>divisible</u> if $M \xrightarrow{a} M$ is surjective for every non-zero $a \in A$.

<u>Examples</u>: 1. Let K be the quotient field of A. Then K is divisible.

2. Let K be as above. Then K/A is divisible.

Prop$^{\underline{n}}$: Let A be a P.I.D. A module ~~E~~ over A is ~~a~~ injective if and ~~module~~ only if it is divisible.

Proof:

Let E be an injective module and $0 \neq a \in A$.

Let $e \in E$. We wish to find $e' \in E$ s.t. $ae' = e$.

Let $I = aA$. Define

$$\varphi : I \longrightarrow E$$

by

$$\varphi(ax) = xe.$$

~~Ex~~ Since E is injective, φ ~~is~~ has an extn $\tilde{\varphi} : A \to E$.

Let $e' = \tilde{\varphi}(1)$.

Then
$$\begin{aligned} ae' &= a\tilde{\varphi}(1) \\ &= \tilde{\varphi}(a \cdot 1) \\ &= \tilde{\varphi}(a) \\ &= \varphi(a) \\ &= e. \end{aligned}$$

Thus E is divisible.

Conversely suppose E is divisible.

Suppose $I \subset A$ is an ideal and we have an ~~map~~ A-map

$$\varphi : I \longrightarrow E.$$

Since A is a P.I.D., $A = aA$ for ~~some~~ $a \in A$.

Let $e = \varphi(a)$. (Note $\varphi(ax) = x\varphi(a) = xe$, $\forall x \in A$).

~~Let $e\ e' \in E$~~ Since E is divisible, there exists $e' \in E$ s.t.

$$e = ae'.$$

Define $\tilde{\varphi} : A \to E$ by $x \mapsto xe'$, $x \in A$. Clearly $\tilde{\varphi}|_I = \varphi$. q.e.d.

~~Examples~~:

From our previous examples we see that $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ etc are injective $\mathbb{Z}$-modules.

In fact $\bigoplus_{i \in I} \mathbb{Q}$ is ~~so~~ divisible, hence injective, for arbitrary non-empty index sets I, ~~So are q~~ as is easy to see. ~~And so~~ This means arbitrary quotients $\bigoplus_{i \in I} \mathbb{Q}/K$ of $\bigoplus_{i \in I} \mathbb{Q}$ are divisible, whence injective as a $\mathbb{Z}$-module.

Since every $\mathbb{Z}$-module is of the form

$$M \cong \bigoplus_{i \in I} \mathbb{Z} / K$$

$(I \neq \phi)$, it follows that every $\mathbb{Z}$-module can be embedded in an injective $\mathbb{Z}$-module. For example ~~take~~ (M as above, embedds in $\bigoplus_{i \in I} \mathbb{Q}/K$).

We have thus proven:

<u>Propn</u>: Every $\mathbb{Z}$-module can be embedded into an injective $\mathbb{Z}$-module.

<u>Lemma</u>: Let $R \to S$ be a map of commutative rings and E an injective R-module. Consider

$$E_S := \mathrm{Hom}_R(S, E),$$

and give it an S-module structure, namely $s\varphi = (s' \mapsto \varphi(ss'))$. Then E_S is an injective S-module.

<u>Proof</u>: Use Hom, $\otimes$ adjointness (Atiyah-MacDonald, Equation (1), page 28; uploaded on website. — need slight generalization of that).

Hom, $\otimes$ adjointness $\Rightarrow$ $\mathrm{Hom}_S(M, E_S) \cong \mathrm{Hom}_A(M \otimes_S S, E) \cong \mathrm{Hom}_A(M, E)$

for every S-module M. Thus we have a functional isomorphism (on Mod_S)

$$\mathrm{Hom}_S(-, E_S) \simeq \mathrm{Hom}_A(-, E).$$

The right side is an exact functor. This forces the left side to be exact also.

q.e.d.

Flat modules:

Let A be a commutative ring, M an A-module. We know that
$$- \otimes_A M : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A$$
is right exact.

Definition: A module M over A is said to be a flat module over A if $- \otimes_A M$ is exact.

Examples:

1. Let $M = A$. Then $- \otimes_A A = 1_{\mathrm{Mod}_A}$, whence A is a flat A-module.
2. Suppose $F = \bigoplus_{i \in I} A$ is a free module. Then
$$T \otimes_A F = \bigoplus_{i \in I} T$$
whence $- \otimes_A F$ is exact. This means free modules are flat.
3. Let P be a projective A-module. Then $\exists$ a free A-module F and a decomposition
$$F = P \oplus Q.$$
$$- \otimes_A F = (- \otimes_A P) \oplus (- \otimes_A Q).$$
It follows that $- \otimes_A P$ and $- \otimes_A Q$ are both exact, whence P is flat. Thus projective modules are flat.

Discussion: According to a problem in HW 4, if M, N are A-modules (A a comm. ring), $F_1^\bullet \to M$ and $F_2^\bullet \to N$ flat resolutions, then
$$H^i(F_1^\bullet \otimes_A N) \simeq H^i(M \otimes F_2^\bullet), \quad \forall\, i \in \mathbb{Z}.$$

In particular, if $F_1^\bullet \to M$ and $G_1^\bullet \to M$ are two flat resolutions then

(*) . . . $H^i(F_1^\bullet \otimes_A N) \simeq H^i(G_1^\bullet \otimes_A N)$ $\forall i \in \mathbb{Z}$
($i \le 0$ enough)

for both sides are ~~isomorphic~~ to $H^i(M \otimes_A F_2^\bullet)$.

In homological algebra, $Tor_i^A(M,N)$ is defined to be $H^{-i}(P^\bullet \otimes_A N)$ where $P^\bullet \to M$ ~~is~~ a projective resolution. This module is independent of the chosen resln $P^\bullet \to M$, since any projective resolution of M is a flat resolution, whence by (*) above ~~we've do~~ $Tor_i^A(M,N)$ is independent of $P^\bullet \to M$.

Moreover, from what we've said, $P^\bullet \to M$, can be replaced by a flat resolution of M.

Finally, from the soln of the HW problem, we see that

$$Tor_i^A(M,N) \simeq Tor_i^A(N,M).$$

~~All these~~ Facts can be summarized as follows

- $Tor_i^A(M,N) := H^{-i}(P^\bullet \otimes_A N)$, $P^\bullet \to M$ ~~resln~~ proj resln of M.
- $Tor_i^A(M,N)$ does not depend on the proj. resln $P^\bullet \to M$ chosen.
- In fact $Tor_i^A(M,N)$ can be computed as $H^{-i}(F^\bullet \otimes N)$ where $F^\bullet \to M$ is a flat resln of M.

  $Tor_i^A(M,N) \simeq Tor_i^A(N,M)$, ~~so~~ i.e., $Tor_i^A(M,N)$ can be computed as $H^{-i}(N \otimes_A G^\bullet)$ where $G^\bullet \to N$ is flat resln of N.

Add to these the fact that $Tor_i^A(F,N)$ is zero for F flat. Indeed $0 \to F \xrightarrow{1} F \to 0$ is a flat resln of F, whence $F^\bullet = F$ can be used for computing $Tor_i^A(F,N) = H^{-i}(F^\bullet \otimes_A N) = \begin{cases} F \otimes_A N, & i = 0 \\ 0 & \text{otherwise} \end{cases}$

All of these facts (together with $Tor_i^A(F,N) = 0$, $i \geq 1$ if F flat) also follow from more general principles involving derived functors

Injective Resolutions and Projective Resolutions:

An abelian category is said to have enough injectives ~~iff~~ (resp. enough projectives) if every object X can be embedded into an injective object (resp. every object X has an epimorphism from a projective object mapping on to it.)

Recall, for the sake of definiteness, a resolution of an object X is a pair $(C^\bullet, c)$ where $C^\bullet$ is a complex with $C^i = 0$ for $i < 0$ (resp. $i > 0$) and ~~a~~ $c: X \to C^\bullet$ is a quasi-isomorphism (resp. $c: C^\bullet \to X$ is a quasi-isomorphism). Often we call the first kind of resolution a right resln and the second kind a left resln.

By an injective resolution of X we mean a right resolution $(C^\bullet, c)$ with C^p injective for all $p \geq 0$

By a projective resolution we mean a left resolution $(C^\bullet, c)$ with C^p projective for all $p \leq 0$.