The Snake Lemma

Some easy facts (check yourself)

1. If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ are two maps s.t. $\beta\alpha : A \to C$ is a monomorphism, then $\alpha : A \to B$ is a monomorphism. If $\beta\alpha$ is an epimorphism, $\beta : B \to C$ is epi.

2. If S is a subgroup of a grp A, $\alpha : A \to A'$ a grp homomorphism, and $\ker\alpha$ is a subgroup of S, then $\ker\alpha = \ker(\alpha|S)$. The generalization of this to exact categories is: Suppose $i : S \to A$ is a monomorphism, $\alpha : A \to A'$ a map, and suppose $\ker\alpha = (K \xrightarrow{j} A)$ factors as a composite $K \xrightarrow{\varphi} S \xrightarrow{i} A$. Then ~~ker~~ $(K \xrightarrow{\varphi} S) = \ker(S \xrightarrow{\alpha i} A)$.

3. To say

$$A \to [\ker(B \to C)] \to 0$$

is exact is to say

$$A \to B \to C$$

is exact. (The two exactness statements are equivalent.)

4. If

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
P \arrow[r] \arrow[d, "\alpha"'] & Q \arrow[r] \arrow[d, "\beta"'] & R \arrow[r] \arrow[d, "\gamma"'] & S \arrow[d, "\delta"] \\
P' \arrow[r] & Q' \arrow[r] & R' \arrow[r] & S'
\end{tikzcd}$$

is an exact commutative diagram then

Should be alpha, beta, and gamma instead of beta, gamma and delta.

$$\operatorname{coker}\beta \to \operatorname{coker}\gamma \to \operatorname{coker}\delta$$

is exact.

Pf: This is the dual of Problem 6 of HW 2.

Should be HW1

Theorem (The Snake Lemma): Suppose we have a commutative diagram with exact rows and columns (i.e. an exact commutative diag.)

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \\
0 & & & &
\end{tikzcd}$$

Then we have a six term exact sequence

$$\ker\beta \to \ker\gamma \to \ker\delta \xrightarrow{\partial} \operatorname{coker}\beta \to \operatorname{coker}\gamma \to \operatorname{coker}\delta$$

where the "connecting homomorphism" ∂ is characterized thus:

① Put $K = \ker(C \to D')$. Then $K \to \ker\delta$ is an epimorphism ;

② Put $K' = \operatorname{coker}(B \to C')$. Then $\operatorname{coker}\beta \to K'$ is a monomorphism ;

③ The two composites :

$$K \to C \xrightarrow{\gamma} C' \to K'$$

and

$$K \xrightarrow{①} \ker\delta \xrightarrow{\partial} \operatorname{coker}\beta \xrightarrow{②} K'$$

are the same.

Remarks

1. ①, ② and ③ indeed characterize the connecting map ∂ for at most one map can fill the dotted arrow to make the diagram below commute:

Connecting homomorphism diagram

$$\begin{tikzcd}
C \arrow[r, "\gamma"] & C' \arrow[d] \\
K \arrow[u] \arrow[d, two heads] & K' \\
\ker\delta \arrow[r, dotted, "\partial"] & \operatorname{coker}\beta \arrow[u, tail]
\end{tikzcd}$$

Here $\twoheadrightarrow$ denotes epimorphism & $\rightarrowtail$ denotes monomorphism.

$$\left[\ i\,\partial\,\pi = i\,\partial'\,\pi\ ,\quad i \text{ mono}, \pi \text{ epi} \ \Rightarrow\ \partial = \partial' \right].$$

$$\begin{tikzcd}
 & & & d \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & {} \arrow[d] \\
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & C' \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & D' \arrow[r] & E' \\
0 & b' & c' & &
\end{tikzcd}$$

Description of the connecting map when we have elements.

2. Suppose all objects were groups and maps group homomorphisms; more precisely suppose we were working with an exact subcategory of the exact category of abelian groups. Then the connecting map has the following description:

Let $d \in \ker \delta$. Then d maps to 0 under $D \to E'$. Now $D \to E'$ factors as $D \to E \xrightarrow{\epsilon} E'$, and since ϵ is a monomorphism, $d \in \ker (D \to E)$. Therefore $\exists\, c \in C$ s.t. c maps to d under $C \to D$. Let $c' = \gamma c$. Clearly c' maps to 0 under $C' \to D'$ (~~being the~~ c' maps to the image of d under δ, and $d \in \ker \delta$). This means there exists $b' \in B'$ which maps to c' under $B' \to C'$. Let $[b'] \in \operatorname{coker} \beta$ denote the image of b' under $B' \to \operatorname{coker} \beta$. Then one checks readily that $[b']$ does not depend on choices made, i.e. on the choice of the preimage c of d, and of the choice of the preimage b' of $c' = \gamma c$. Then

$$\partial(d) = [b'].$$

In fact, the element $c \in K = \ker (C \to D')$, and the image of γc in ~~K', under~~ $K' := \operatorname{coker} (B \to C')$ is precisely the image of $[b']$ in K' under the natural map $\operatorname{coker} \beta \to K'$. Thus ~~$d \mapsto [b']$ coker~~ $\ker \delta \to \operatorname{coker} \beta$, $d \mapsto [b']$ fills the dotted arrow in the diagram for connecting maps. This means $d \mapsto [b']$, $d \in \ker \delta$ is the connecting map.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] & C' \arrow[r] \arrow[dr, "f"'] & D' \arrow[r] & E' \\
0 & & & \ker(D' \to E') \arrow[u] &
\end{tikzcd}$$

exact comm diag $\Big\} \Rightarrow \ker\beta \to \ker\gamma \to \ker\delta \overset{?}{\to} \operatorname{coker}\beta \to \operatorname{coker}\gamma \to \operatorname{coker}\delta$ exact.

HW1

<u>Proof of the Snake Lemma</u>: By problem 6 of HW2, and its dual (see Remark 4 before statement of Thm) we already have

$$\ker\beta \to \ker\gamma \to \ker\delta \quad \text{(exact)}$$

&

$$\operatorname{coker}\beta \to \operatorname{coker}\gamma \to \operatorname{coker}\delta \quad \text{(exact)}.$$

Let us prove ① and ② in the statement. Let the natural map $C' \to \ker(D' \to E')$ be denoted f. Consider the exact commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \operatorname{im}(f) \arrow[r] & D' \arrow[r] & E' \\
0 & 0 & & &
\end{tikzcd}$$

Apply Problem 6, HW2 ~~to~~ Twice to get that

$$(*) \qquad B \to K \to \ker\delta \to 0$$

is exact. This proves ①.

To prove ② use the exact commutative diagram (with g the map ~~[illegible]~~ $\operatorname{coker}(A \to B) \to C$)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] \\
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"] & \operatorname{im} g \arrow[r] \arrow[d, "\gamma g"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \\
0 & & & &
\end{tikzcd}$$

Should be "gamma restricted to im(g)" instead of "gamma composed with g" for the map im(g)--> C'.

and apply Rmk 4 before the statement of the Snake Lemma (apply it twice) to get an exact sequence

$$(**) \qquad 0 \to \operatorname{coker}\beta \to K' \to D'$$

Thus ② is also true.

Now the composite

$$B \to K \to K'$$

is zero, where the second arrow is $K \hookrightarrow C \xrightarrow{\gamma} C' \twoheadrightarrow K'$. Indeed the map $B \to K'$ above is the same as the composite $B \to C' \to \operatorname{coker}(B \to C')(= K') \to 0$. By $(*)$ $\ker\delta = \operatorname{coker}(B \to K)$, whence we get a unique map

$$(+) \qquad \ker\delta \to K'$$

such that the composite $K \hookrightarrow C \xrightarrow{\gamma} C' \twoheadrightarrow K'$ is the same as the composite

$$K \underset{\uparrow \text{ from } (*)}{\longrightarrow} \ker\delta \xrightarrow[(+)]{} K' .$$

Now $K \to K' \to D'$ is ~~zero~~ zero, therefore, since $K \to \ker\delta$ is an epimorphism (see $(*)$), the composite

$$\ker\delta \xrightarrow{(+)} K' \to D'$$

is zero. Now according to $(**)$, $\operatorname{coker}\beta = \ker(K' \to D')$, and hence we have a unique map

$$\partial : \ker\delta \to \operatorname{coker}\beta$$

such that $(+)$ is the composite $\ker\delta \xrightarrow{\partial} \operatorname{coker}\beta \to K'$. Thus the connecting homomorphism as charaterized by ①, ② and ③ has been shown to exist. It remains to show that it fits into ~~an~~ the asserted exact sequence. It clearly suffices to prove the exactness of

$$\ker\gamma \to \ker\delta \xrightarrow{(+)} K' .$$

For that consider the exact commutative diagram:

$$
\begin{tikzcd}
B \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & \ker\delta \arrow[r] \arrow[d, "(+)"] & 0 \arrow[d, equal] \\
B \arrow[r] \arrow[d] & C' \arrow[r] & K' \arrow[r] & 0 \\
0 & & \overset{\shortparallel}{\operatorname{coker}(B \to C')} &
\end{tikzcd}
$$

where the unlabelled arrow (e.g. $K \to C'$) are obvious. ~~composites, or~~ By Problem 6, HW1, we get an exact sequence

$$\ker(K \to C') \longrightarrow \ker(+) \longrightarrow 0.$$

Using Rmk 2 before statement of Thm, we see that $\ker(K \to C') = \ker\gamma$. Using Rmk 3 before the statement of Thm, we see that

$$\ker\gamma \longrightarrow \operatorname{coker}\delta \xrightarrow{(+)} K'$$

is exact. This proves the Snake Lemma.