

The Snake Lemma

Some easy facts (check yourself)

1. If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ are two maps s.t. $\beta\alpha: A \rightarrow C$ is a monomorphism, then $\alpha: A \rightarrow B$ is a monomorphism. If $\beta\alpha$ is an epimorphism, $\beta: B \rightarrow C$ is epi.

2. If S is a subgroup of a grp A , $\alpha: A \rightarrow A'$ a grp homomorphism, and $\ker \alpha$ is a subgroup of S , then $\ker \alpha = \ker(\alpha|_S)$. The generalization of this to exact categories is: Suppose $i: S \rightarrow A$ is a monomorphism, $d: A \rightarrow A'$ a map, and suppose $\ker d = (K \xrightarrow{j} A)$ factors as a composite $K \xrightarrow{\phi} S \xrightarrow{i} A$. Then $\ker(K \xrightarrow{\phi} S) = \ker(S \xrightarrow{\alpha i} A)$.

3. To say

$$A \rightarrow [\ker(B \rightarrow C)] \rightarrow 0$$

is exact is to say

$$A \rightarrow B \rightarrow C$$

is exact. (The two exactness statements are equivalent.)

4. If

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & S \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \downarrow \delta \\ P' & \longrightarrow & Q' & \longrightarrow & R' & \longrightarrow & S' \end{array}$$

is an exact commutative diagram then

$$\operatorname{coker} \beta \rightarrow \operatorname{coker} \gamma \rightarrow \operatorname{coker} \delta$$

is exact.

Pf: This is the dual of Problem 6 of HW 2.

Should be alpha, beta, and gamma instead of beta, gamma and delta.

Should be HW1

Theorem (The Snake Lemma): Suppose we have a commutative diagram with exact rows and columns (i.e. an exact commutative diag.)

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
 \downarrow & & & & & & & & \downarrow \\
 & & & & & & & & 0
 \end{array}$$

Then we have a six term exact sequence

$$\ker \beta \rightarrow \ker \gamma \rightarrow \ker \delta \xrightarrow{\partial} \operatorname{coker} \beta \rightarrow \operatorname{coker} \gamma \rightarrow \operatorname{coker} \delta$$

where the "connecting homomorphism" ∂ is characterized thus:

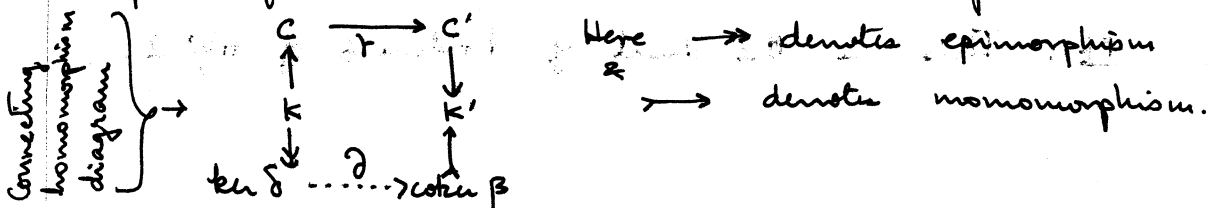
- ① Put $K = \ker(C \rightarrow D')$. Then $K \rightarrow \ker \delta$ is an epimorphism;
- ② Put $K' = \operatorname{coker}(B \rightarrow C')$. Then $\operatorname{coker} \beta \rightarrow K'$ is a monomorphism;
- ③ The two composites:

$$\begin{array}{l}
 \text{and} \\
 K \xrightarrow{\gamma} C \xrightarrow{\gamma} C' \rightarrow K' \\
 K \xrightarrow{\text{①}} \ker \delta \xrightarrow{\partial} \operatorname{coker} \beta \xrightarrow{\text{②}} K'
 \end{array}$$

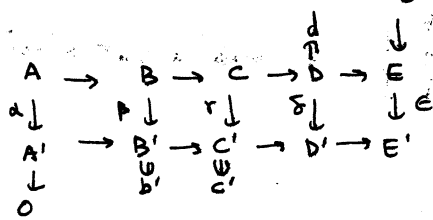
are the same.

Remarks

1. ①, ② and ③ indeed characterize the connecting map ∂ for at most one map can fill the dotted arrows to make the diagram below commute:



$$\left[i \partial \pi = i \partial' \pi, i \text{ mono}, \pi \text{ epi} \Rightarrow \partial = \partial' \right].$$



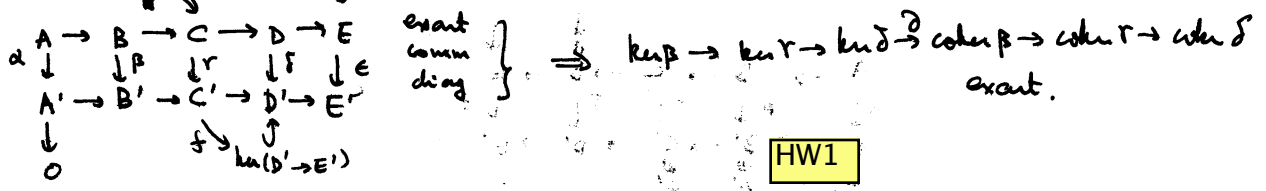
Description of the connecting map when we have elements.

2. Suppose all objects were groups and maps group homomorphisms; more precisely suppose we were working with an exact subcategory of the category of abelian groups. Then the connecting map has the following description:

Let $d \in \ker \delta$. Then d maps to 0 under $D \rightarrow E'$.
 Now $D \rightarrow E'$ factors as $D \rightarrow E \xrightarrow{\epsilon} E'$, and since ϵ is a monomorphism, $d \in \ker(D \rightarrow E)$. Therefore $\exists c \in C$ s.t. c maps to d under $C \rightarrow D$. Let $c' = \gamma c$. Clearly c' maps to 0 under $C' \rightarrow D'$ (being the c' maps to the image of d under δ , and $d \in \ker \delta$). This means there exists $b' \in B'$ which maps to c' under $B' \rightarrow C'$. Let $[b'] \in \text{coker } \beta$ denote the image of b' under $B' \rightarrow \text{coker } \beta$. Then one checks readily that $[b']$ does not depend on choices made, i.e. on the choice of the preimage c of d , and of the choice of the preimage b' of $c' = \gamma c$. Then

$$\partial(d) = [b'].$$

In fact, the element $c \in K = \ker(C \rightarrow D')$, and the image of γc in $K' = \text{coker}(B \rightarrow C')$ is precisely the image of $[b']$ in K' under the natural map $\text{coker } \beta \rightarrow K'$. Thus $d \mapsto [b'] \in \text{coker } \beta$ fills the dotted arrow in the diagram for connecting maps. This means $d \mapsto [b']$, $d \in \ker \delta$ is the connecting map.



HW1

Proof of the Snake Lemma: By problem 6 of HW2, and its dual

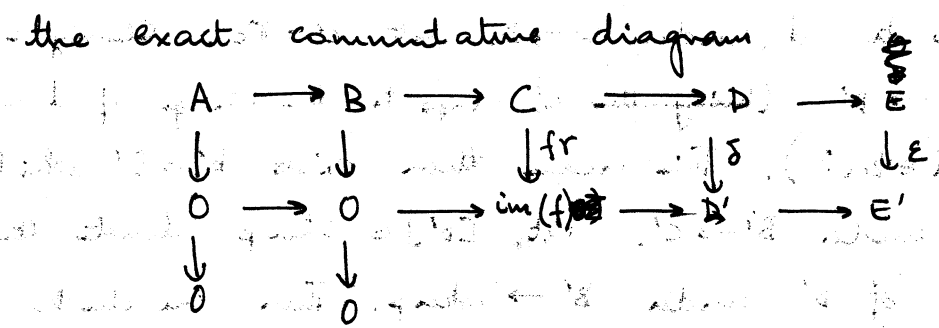
(see Remark 4 before statement of Thm) we already have

$$\ker \beta \rightarrow \ker \gamma \rightarrow \ker \delta \quad (\text{exact})$$

&

$$\text{coker } \beta \rightarrow \text{coker } \gamma \rightarrow \text{coker } \delta \quad (\text{exact}).$$

Let us prove ① and ② in the statement. Let the natural map $(C' \rightarrow \ker(D' \rightarrow E'))$ be denoted f . Consider

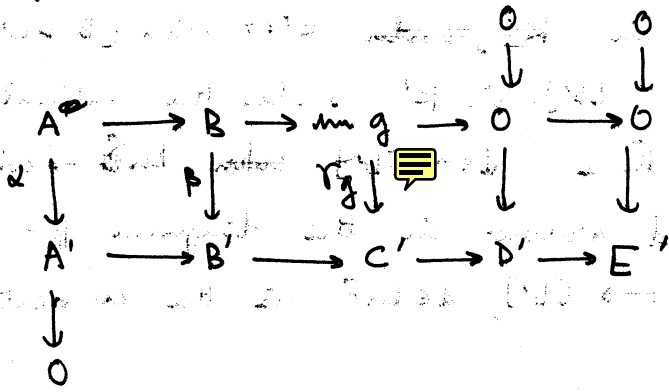


Apply Problem 6, HW2 twice to get that

(*) $B \rightarrow K \rightarrow \ker \delta \rightarrow D$

is exact. This proves ①.

To prove ② use the exact commutative diagram (with g the map $\text{coker}(A \rightarrow B) \rightarrow C$)



Should be "gamma restricted to im(g)" instead of "gamma composed with g" for the map $\text{im}(g) \rightarrow C'$.

and apply Remark 4 before the statement of the Snake Lemma (apply it twice) to get an exact sequence

(**) $0 \rightarrow \text{coker } \beta \rightarrow K' \rightarrow D'$

Thus ② is also true.

Now the composite

$$B \rightarrow K \rightarrow K'$$

is zero, where the second arrow is $K \hookrightarrow C \xrightarrow{\gamma} C' \rightarrow K'$. Indeed

the map $B \rightarrow K'$ above is the same as the composite

$$B \rightarrow C' \rightarrow \text{coker}(B \rightarrow C')(K') \rightarrow D. \quad \text{By } (*) \text{ } \ker \delta = \text{coker}(B \rightarrow K),$$

whence we get a unique map

$$(+): \quad \ker \delta \rightarrow K'$$

such that the composite $K \hookrightarrow C \xrightarrow{\gamma} C' \rightarrow K'$ is the same as the composite

$$\begin{array}{ccc} K & \xrightarrow{\quad} & \ker \delta \xrightarrow{(+)} K' \\ & \nearrow \text{from } (*) & \nwarrow (+) \end{array}$$

Now $K \rightarrow K' \rightarrow D'$ is zero, therefore, since $K \rightarrow \ker \delta$ is an epimorphism (see $(*)$), the composite

$$\ker \delta \xrightarrow{(+)} K' \rightarrow D'$$

is zero. Now according to $(**)$, $\text{coker } \beta = \ker(K' \rightarrow D')$, and hence we have a unique map

$$\partial: \ker \delta \rightarrow \text{coker } \beta$$

such that $(+)$ is the composite $\ker \delta \xrightarrow{\partial} \text{coker } \beta \rightarrow K'$.

Thus the connecting homomorphism as characterized by ①, ② and ③ has been shown to exist. It remains to show that it fits into ~~an~~ ^{the asserted} exact sequence. It clearly suffices to prove the exactness of

$$\ker r \rightarrow \ker \delta \xrightarrow{(+)} K'$$

For that consider the exact commutative diagram:

$$\begin{array}{ccccccc}
 B & \longrightarrow & K & \longrightarrow & \ker \delta & \longrightarrow & 0 \\
 \parallel & & \downarrow & & \downarrow (+) & & \parallel \\
 B & \longrightarrow & C' & \longrightarrow & K' & \longrightarrow & 0 \\
 \downarrow & & & & \text{"coker}(B \rightarrow C') & & \\
 0 & & & & & &
 \end{array}$$

where the unlabelled arrows (e.g. $K \rightarrow C'$) are obvious.

~~composites~~ By Problem 6, HW1, we get an exact sequence

$$\ker(K \rightarrow C') \longrightarrow \ker(+)\longrightarrow 0.$$

Using Rank 2 before statement of Thm, we see that $\ker(K \rightarrow C') = \ker \gamma$. Using Rank 3 before the statement

of Thm, we see that

$$\ker \gamma \longrightarrow \text{coker } \delta \xrightarrow{(+)} K'$$

is exact. This proves the Snake Lemma.