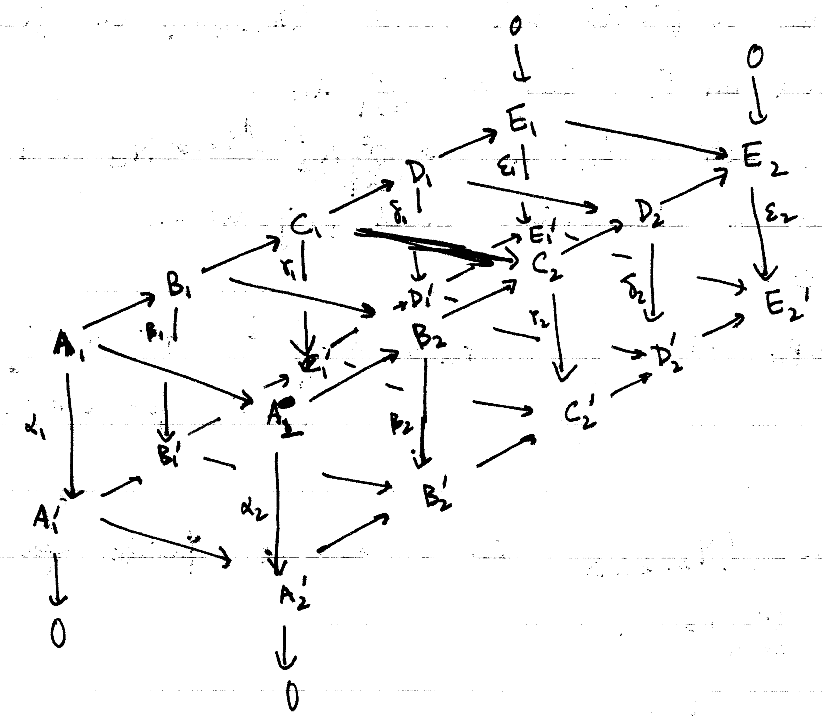
y 23, 2010

Homotopy Theory of complexes

§ Functoriality of the connecting homomorphism :



Exact commutative diag.

$$\Rightarrow\quad\begin{tikzcd}
\ker\beta_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \ker\gamma_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \ker\delta_1 \arrow[r, "\partial"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{coker}\beta_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{coker}\gamma_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{coker}\delta_1 \arrow[d] \\
\ker\beta_2 \arrow[r] & \ker\gamma_2 \arrow[r] & \ker\delta_2 \arrow[r, "\partial"'] & \operatorname{coker}\beta_2 \arrow[r] & \operatorname{coker}\gamma_2 \arrow[r] & \operatorname{coker}\delta_2
\end{tikzcd}$$

Follows from

$$\begin{tikzcd}
K_1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_1 \arrow[r, "\gamma_1"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_1' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & K_1' \arrow[d] \\
K_2 \arrow[r, hook] & C_2 \arrow[r, "\gamma_2"] & C_2' \arrow[r] & K_2'
\end{tikzcd}$$

We work throughout in an abelian category.

Matrix notation:

Let T be a finite set and $(X_t)_{t\in T}$ a family of objects indexed by T. Let $X = \bigoplus_{t\in T} X_t$.
For each t, let $i_t : X_t \to X$ and $X \to X_t$ be the standard inclusion and projection.
Note for $u, v \in T$,

$$\sum_{t\in T} i_t p_t = 1 \qquad p_u i_v = \begin{cases} 1 & \text{for } u=v \\ 0 & \text{for } u \neq v. \end{cases}$$

Now suppose $(Y_s)_{s\in S}$ is a second finite family of objects, $Y = \bigoplus_{s\in S} Y_s$.

To a morphism

$$f : X \to Y$$

we associate a *matrix*

$$(f_{st})_{(s,t)\in S\times T} \; ; \; f_{st} = p_s f i_t : X_t \to Y_s.$$

Conversely, to such a matrix with

$$f_{st} \in \mathrm{Hom}(X_t, Y_s)$$

there exists a uniquely determined morphism $f : X \to Y$.
Now suppose $Z = \bigoplus_{r\in R} Z_r$, R a finite set. For morphisms $f : X \to Y$ and $g : Y \to Z$, we have

$$(gf)_{rt} = \sum_s g_{rs} f_{st}.$$

Example: $X = X_1 \oplus X_2$. The standard inclusions $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ have matrices $\binom{1_{X_1}}{0}$ and $\binom{0}{1_{X_2}}$, whereas the standard projections $X \xrightarrow{p_1} X_1$, $X \xrightarrow{p_2} X_2$ have matrices $(1_{X_1}, 0)$ & $(0, 1_{X_2})$

Defn: A complex $C^\bullet = (C^n, \partial^n)$ is a sequence of objects and morphisms

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \longrightarrow \cdots$$

with $\partial^n \partial^{n-1} = 0 \quad \forall n \in \mathbb{Z}$.

no dots

A <u>morphism of complexes</u> $f^\bullet : C^\bullet \to D^\bullet$ $\quad (f^\bullet = (f^n)_{n\in\mathbb{Z}})$ is a sequence of maps $f^n : C^n \to D^n$ with

$$f^{n+1}\partial^n = \partial^n f^n.$$

$$\begin{tikzcd} \cdots \arrow[r] & C^n \arrow[r, "\partial^n"] \arrow[d, "f^n"'] \arrow[dr, phantom, "\circlearrowright"] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ \cdots \arrow[r] & D^n \arrow[r, "\partial^n"'] & D^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

Defn: $C^\bullet$ a complex, $n \in \mathbb{Z}$.

$$H^n(C^\bullet) = \ker \partial^n / \operatorname{im}(\partial^{n-1})$$

nth cohomology of $C^\bullet$.

Say nth cohomology is a functor from complexes to the underlying category.

Notation: For any complex $A^\bullet$ put $Z^m(A^\bullet) = \ker \partial^m$, and ${}'Z^m(A^\bullet) = \operatorname{coker}(\partial^{m-1})$.

$$0 \to Z^m(A^\bullet) \longrightarrow A^m \longrightarrow A^{m+1} \qquad \text{(exact)}$$

$$A^{m-1} \longrightarrow A^m \longrightarrow {}'Z^m(A^\bullet) \longrightarrow 0 \qquad \text{(exact)}.$$

Definition: A sequence of complexes

$$\cdots \to C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to \cdots$$

is said to be exact if, for each $n \in \mathbb{Z}$,

$$\cdots \to C_1^n \to C_2^n \to C_3^n \to \cdots$$

is exact.

Proposition: Given an exact sequence of complexes

$$0 \to C^\bullet \to D^\bullet \to E^\bullet \to 0$$

there exists a canonical exact seq.

$$\cdots \to H^n(D^\bullet) \to H^n(E^\bullet) \xrightarrow{c} H^{n+1}(C^\bullet) \to H^{n+1}(D^\bullet) \to H^{n+1}(E^\bullet) \to \cdots$$

Moreover the connecting map c is functorial, i.e., if

$$\begin{tikzcd} 0 \arrow[r] & C_1^\bullet \arrow[r] \arrow[d] & D_1^\bullet \arrow[r] \arrow[d] & E_1^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_2^\bullet \arrow[r] & D_2^\bullet \arrow[r] & E_2^\bullet \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of complexes with each row being an exact sequence of complexes, then

$$\text{(Conn).} \quad \begin{tikzcd} H^n(E_1^\bullet) \arrow[r, "c"] \arrow[d] & H^{n+1}(C_1^\bullet) \arrow[d] \\ H^n(E_2^\bullet) \arrow[r, "c"'] & H^{n+1}(C_2^\bullet) \end{tikzcd}$$

Commutes where c in either row is the connecting homomorphism.

Proof:

For any complex $A^\bullet$, we have a canonical factorization of ∂^n $(n \in \mathbb{Z})$

$$A^n \to {}'Z^n(A^\bullet) \to Z^{n+1}(A^\bullet) \to A^{n+1}$$

Now

$$H^n(A^\bullet) = \ker({}'Z^n(A^\bullet) \to Z^{n+1}(A^\bullet))$$

and

$$H^{n+1}(A^\bullet) = \operatorname{coker}({}'Z^n(A^\bullet) \to Z^{n+1}(A^\bullet)).$$

Fix $n \in \mathbb{Z}$

$$\begin{array}{ccccccccc} 0 & \to & C^{n-1} & \to & D^{n-1} & \to & E^{n-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^n & \to & D^n & \to & E^n & \to & 0 \end{array}$$

implies the exactness of

$$\left.\begin{array}{l} {}'Z^n(C^\cdot) \to {}'Z^n(D^\cdot) \to {}'Z^n(E^\cdot) \to 0 \\ \text{and of} \\ 0 \to Z^{n-1}(C^\cdot) \to Z^{n-1}(D^\cdot) \to Z^{n-1}(E^\cdot) \end{array}\right\} \text{Snake lemma}$$

One therefore has a comm. diag with exact rows

$$\begin{array}{ccccccc} & {}'Z^n(C^\cdot) & \to & {}'Z^n(D^\cdot) & \to & {}'Z^n(E^\cdot) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & Z^{n+1}(C^\cdot) & \to & Z^{n+1}(D^\cdot) & \to & Z^{n+1}(E^\cdot) & \end{array}$$

Apply the Snake Lemma to get the conclusion.

Commutativity of (Conn) follows from the functoriality of the connecting map in the Snake Lemma. QED.

Remarks:

1. If $C^\cdot \xrightarrow{f} D^\cdot$ is a map of complexes, the induced map
$H^n(f) : H^n(C^\cdot) \longrightarrow H^n(D^\cdot)$
is the one which makes the diagram (For any cmplx $j : Z^n(A^\cdot) \to A^n$ is the canonical monomorphism).

$$\begin{array}{ccc} C^n & \xrightarrow{f^n} & D^n \\ j \uparrow & & \uparrow j \\ Z^n(C^\cdot) & \xrightarrow{f^n} & Z^n(D^\cdot) \\ \pi \downarrow & & \downarrow \pi \\ H^n(C^\cdot) & \xrightarrow[H^n(f)]{} & H^n(D^\cdot) \end{array}$$

Commute, where for any cmplx $A^\cdot$, π is the canonical map $C^n \longrightarrow \mathrm{coker}(\partial^{n-1}) = C^n/_{\mathrm{im}\,\partial^{n-1}} = {}'Z^n(C^\cdot)$. More precisely the following cube commutes and defines $H^n(f)$:

$$
\begin{tikzcd}
 & D^n \arrow[dd, two heads, "\pi"] & & Z^n(D^\bullet) \arrow[ll, hook', "j"'] \arrow[dd, two heads, "\pi"] \\
C^n \arrow[ur, "f"] \arrow[dd, two heads, "\pi"'] & & Z^n(C^\bullet) \arrow[ll, hook', "j"'] \arrow[ur, "f"] \arrow[dd, two heads, "\pi"] & \\
 & {}^{\backprime}Z^n(D^\bullet) & & H^n(D^\bullet) \arrow[ll, hook', "\text{via } j"] \\
{}^{\backprime}Z^n(C^\bullet) \arrow[ur, "\text{via coker}\,\pi\text{'s}"] & & H^n(C^\bullet) \arrow[ll, hook', "\text{via } j"] \arrow[ur, "H^n(f)"'] &
\end{tikzcd}
$$

In particular, the composite

induced by j

$$Z^n(C^\bullet) \xrightarrow{\pi} H^n(C^\bullet) \xrightarrow{H^n(f)} H^n(D^\bullet) \hookrightarrow {}^{\backprime}Z^n(D^\bullet)$$

is also the composite

$$Z^n(C^\bullet) \xhookrightarrow{j} C^n \xrightarrow{f} D^n \xrightarrow{\pi} {}^{\backprime}Z^n(D^\bullet).$$

2. We can form a category of short exact sequences of complexes, and another category consisting of long exact sequence of objects. Then what we have shown is that there is a functor from one to the other.

Trying to switch to Mondays Fridays. First Friday class.

Std. Notation: $Z^n(A^\cdot) = \ker(A^n \xrightarrow{\partial^n} A^{n+1})$
$'Z^n(A^\cdot) = \mathrm{coker}(A^{n-1} \xrightarrow{\partial^{n-1}} A^n)$.

Everything happening in an abelian cat.

27.8.10.

<u>Homotopy</u>: Let $f, g : C^\cdot \to D^\cdot$ be two maps of complexes. A <u>homotopy from f to g</u> is a sequence $s = (s^n)_{n\in\mathbb{Z}}$ of maps $s^n : C^n \to D^{n-1}$ such that

$$f^n - g^n = \partial^{n-1} s^n + s^{n+1}\partial^n, \quad n \in \mathbb{Z}.$$

If this is so, we write $f \sim g$.

<u>Proposition</u>: Let $f, g : C^\cdot \to D^\cdot$ be homotopic maps. Then

$$H^n(f) = H^n(g), \quad n \in \mathbb{Z}.$$

<u>Proof</u>: Fix $n \in \mathbb{Z}$.

We have remarked earlier that the composite (for $\alpha : C^\cdot \to D^\cdot$ map of cplxes)

① $$Z^n(C^\cdot) \xrightarrow{\pi} H^n(C^\cdot) \xrightarrow{H^n(\alpha)} H^n(D^\cdot) \xhookrightarrow{\text{via } j} {}'Z^n(D^\cdot)$$

is the composite

② $$Z^n(C^\cdot) \xhookrightarrow{j} C^n \xrightarrow{\alpha} D^n \xrightarrow{\pi} {}'Z^n(D^\cdot).$$

As earlier, if $A^\cdot$ is a complex, $j : Z^n(A^\cdot) \hookrightarrow A^n$ nat'l mono., & $\pi : A^n \to {}'Z^n(A^\cdot)$ nat'l epi.

Since $\pi : Z^n(C^\cdot) \to H^n(C^\cdot)$ is an epimorphism and the map $H^n(D^\cdot) \to {}'Z^n(D^\cdot)$ induced by j is a monomorphism, the Prop^n. is proved if we show that substituting $\alpha = f$ and $\alpha = g$ in ① give the same composites. This then amounts to showing that substituting $\alpha = f$ and $\alpha = g$ in ② give the same composite in ②.

Now,

$$\begin{aligned}(f^n - g^n)j &= (\partial^{n-1} s_n - s^{n+1}\partial^n)j \\ &= \partial^{n-1} s_n j - s^{n+1}\partial^n j \\ &= \partial^{n-1} s_n j \qquad (\text{since } \partial^n \circ j = 0)\end{aligned}$$

Therefore

$$\begin{aligned}\pi \circ (f^n - g^n) \circ j &= \pi \circ \partial^{n-1} \circ s_n \circ j \\ &= 0 \qquad (\text{since } \pi \circ \partial^{n-1} = 0, \ \pi \text{ being the cokernel of } \partial^{n-1})\end{aligned}$$

$$\Rightarrow \quad \pi \circ f^n \circ j = \pi \circ g^n \circ j$$

q.e.d.

Notation: $A^\bullet$ complex, $A^\bullet[m] = \left(\cdots \xrightarrow{(-1)^m\partial} \underset{n\text{th spot}}{A^{n+m}} \xrightarrow{(-1)^m\partial} \underset{(n+1)^{\text{th}}\text{spot}}{A^{n+m+1}} \xrightarrow{(-1)^m\partial} \underset{(n+2)^{\text{th}}\text{spot}}{A^{n+m+2}} \to \cdots \right)$

$\left((A^\bullet[m])^n = A^{n+m}; \ \& \ \ \partial^n_{A^\bullet[m]} = (-1)^m \partial^{m+n}_{A^\bullet} \right)$.

Mapping Cones:

Let $f: P^\bullet \to Q^\bullet$ be a map of complexes.

Define a complex

$$C^\bullet_u = C^\bullet_u$$

by the following rules

$$C^n = P^{n+1} \oplus Q^n$$

and

$$\partial^n_{C^\bullet} = \begin{pmatrix} -\partial^{n+1}_{P^\bullet} & 0 \\ f & \partial^n_{Q^\bullet} \end{pmatrix} : \underset{C^n}{\underset{\parallel}{P^{n+1} \oplus Q^n}} \longrightarrow \underset{C^{n+1}}{\underset{\parallel}{P^{n+2} \oplus Q^{n+1}}}.$$

Let $i: Q^\bullet \to C^\bullet$ be the map given at each level n by $i^n: Q^n \to P^{n+1} \oplus Q^n$ (the nat'l inclusion) and $p: C^\bullet \to P^\bullet[1]$ be given by the projection $p^n: P^{n+1} \oplus Q^n \to P^{n+1}$. One checks easily that i and p are maps of complexes. Indeed ~~consider~~, note that $i^n = \begin{pmatrix} 0 \\ 1_{Q^n} \end{pmatrix}$ and $p^n = (1_{P^{n+1}}, 0)$ in matrix ~~notation~~ notation.

We have to check

(a) $$\partial^n_{C^\bullet} i^n = i^{n+1} \partial^n_{Q^\bullet}$$

and (b) $$\partial^n_{P^\bullet[1]} p^n = p^{n+1} \partial^n_{C^\bullet}$$

i.e. $$-\partial^{n+1}_{P^\bullet} \circ p^n = p^{n+1} \circ \partial^n_{C^\bullet}.$$

Then (a) is the matrix equality

$$\begin{pmatrix} -\partial^{n+1}_{P^\bullet} & 0 \\ f & \partial^n_{Q^\bullet} \end{pmatrix} \begin{pmatrix} 0 \\ 1_{Q^n} \end{pmatrix} = \begin{pmatrix} 0 \\ 1_{Q^{n+1}} \end{pmatrix} \partial^n_{Q^\bullet}$$

(Both sides equal $\begin{pmatrix} 0 \\ \partial^n_{Q^\bullet} \end{pmatrix}$.)

Similarly (b) is verified by checking

$$-\partial^{n+1}_{P^\bullet}\;(1_{P^{n+1}}, 0) = (1_{P^{n+2}}, 0)\begin{pmatrix} -\partial^{n+1}_{P^\bullet} & 0 \\ f & \partial^n_{Q^\bullet} \end{pmatrix}.$$

In fact both sides equal $(-\partial^{n+1}_{P^\bullet}, 0)$.

<u>Lemma</u>: We have an exact sequence of complexes

$$(*) \qquad 0 \longrightarrow Q^\bullet \xrightarrow{i} C^\bullet_f \xrightarrow{p} P^\bullet[1] \longrightarrow 0$$

where $C^\bullet_f$ is the mapping cone of $f: P^\bullet \to Q^\bullet$ and for each n, in matrix notation.

$$i^n = \begin{pmatrix} 0 \\ 1_{Q^n} \end{pmatrix} \quad \text{and} \quad p^n = (1_{P^{n+1}}, 0).$$

//

~~Proposition~~

In what follows we fix the map $f: P^\bullet \to Q^\bullet$ above, and use the notations and notions introduced above.

<u>Proposition</u>: Fix $n \in \mathbb{Z}$ and consider the commutative exact diagram arising from $(*)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q^{n-1} & \xrightarrow{i^{n-1}} & C^{n-1}_f & \xrightarrow{p^{n-1}} & P^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^{n-1}} & & \downarrow{\scriptstyle \partial^{n-1}_{C^\bullet}} & & \downarrow{\scriptstyle -\partial^n} & & \\
0 & \longrightarrow & Q^n & \longrightarrow & C^n_f & \longrightarrow & P^{n+1} & \longrightarrow & 0
\end{array}$$

Then the resulting connecting map

$$\varphi: Z^n(P^\bullet) \longrightarrow {}^{`}Z^n(Q^\bullet)$$

is the composite

$$Z^n(P^\bullet) \xrightarrow{\pi} H^n(P^\bullet) \xrightarrow{H^n(f)} H^n(Q^\bullet) \xhookrightarrow{\text{via } j} {}^{`}Z^n(Q^\bullet).$$

Proof: We continue use our notations:

$$j : Z^n(A^\bullet) \hookrightarrow A^n$$

and

$$\pi : \cancel{Z} A^n \twoheadrightarrow {}'Z^n(A^\bullet)$$

for every complex $A^\bullet$ and every integer n.

Let

$$K = \ker(C_f^{n+1} \longrightarrow P^{n+1})$$

and

$$K' = \operatorname{coker}(Q^{n-1} \longrightarrow C_f^n).$$

Now $C_f^{n-1} \longrightarrow P^{n+1}$ is the map $(-\partial^n_{P^\bullet}, 0)$ and $Q^{n-1} \longrightarrow C_f^n$ is the map $\begin{pmatrix} 0 \\ \partial^{n-1}_{C^\bullet} \end{pmatrix}$. Therefore

$$K = Z^n(P^\bullet) \oplus Q^{n-1}$$

and

$$K' = P^{n+1} \oplus {}'Z^n(Q^\bullet).$$

Moreover the natural maps

$$u : K \hookrightarrow C^{n-1}$$

$$v : C^n \twoheadrightarrow K'$$

are given by

$$u = \begin{pmatrix} j & 0 \\ 0 & 1_{Q^{n-1}} \end{pmatrix} : Z^n(P^\bullet) \oplus Q^{n-1} \longrightarrow P^n \oplus Q^{n-1}$$

and

$$v = \begin{pmatrix} 1_{P^{n+1}} & 0 \\ 0 & \pi \end{pmatrix} : P^{n+1} \oplus Q^n \longrightarrow P^{n+1} \oplus {}'Z^n(Q^\bullet).$$

The composite

$$\begin{array}{ccccccc} K & \xrightarrow{u} & C^{n-1} & \xrightarrow{\partial^{n-1}_{C^\cdot}} & C^n & \xrightarrow{v} & K' \\ \parallel & & & & & & \parallel \\ Z^n(P^\cdot)\oplus Q^{n-1} & & & & & & P^{n+1}\oplus {}^{\backprime}Z^n(Q^\cdot) \end{array}$$

is therefore given by the matrix ~~product~~

$$\begin{pmatrix} 1_{P^{n+1}} & 0 \\ 0 & \pi \end{pmatrix}\begin{pmatrix} -\partial^n_{P^\cdot} & 0 \\ f^n & \partial^{n-1}_{Q^\cdot} \end{pmatrix}\begin{pmatrix} j & 0 \\ 0 & 1_{Q^{n-1}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \pi f^n j & 0 \end{pmatrix}$$

If $\alpha : K \twoheadrightarrow Z^n(P^\cdot)$ is the natural map (induced by $p^{n-1} : C^{n-1} \to P^n$) and $\beta : {}^{\backprime}Z^n(Q^\cdot) \hookrightarrow K'$ the natural map induced by $i^n : Q^n \to P^{n+1}\oplus Q^n$, then by the construction of the connecting map, the diagram

$$\begin{array}{ccc} Z^n(P^\cdot)\oplus Q^{n-1} = K & \xrightarrow{v\circ\partial^{n-1}_{C^\cdot}\circ u} & K' = P^{n+1}\oplus {}^{\backprime}Z^n(Q^\cdot) \\ \alpha\downarrow & & \uparrow\beta \\ Z^n(P^\cdot) & \xrightarrow[\varphi]{} & {}^{\backprime}Z^n(Q^\cdot) \end{array}$$

commutes, and φ is the only ~~diagram~~ map which makes the diagram commute, for α is epi and β mono. Now

$$\alpha = (1_{Z^n(P^\cdot)}, 0) \quad \text{and} \quad \beta = \begin{pmatrix} 0 \\ 1_{{}^{\backprime}Z^n(Q^\cdot)} \end{pmatrix}.$$

Moreover the diagram below clearly commutes (basic matrix multiplication).

$$\begin{array}{ccc} Z^n(P^\cdot)\oplus Q^{n-1} & \xrightarrow{\begin{pmatrix} 0 & 0 \\ \pi f^n j & 0 \end{pmatrix} = v\,\partial^{n-1}_{C^\cdot}\,u} & P^{n+1}\oplus {}^{\backprime}Z^n(Q^\cdot) \\ \alpha = (1_{Z^n(P^\cdot)}, 0)\Big\downarrow & & \Big\uparrow \beta = \begin{pmatrix} 0 \\ 1_{{}^{\backprime}Z^n(Q^\cdot)} \end{pmatrix} \\ Z^n(P^\cdot) & \xrightarrow[\pi f^n j]{} & {}^{\backprime}Z^n(Q^\cdot) \end{array}$$

Therefore, $\varphi = \pi\circ f^n\circ j$. On the other hand by an earlier

Remark the composite

$$Z^n(P^\cdot) \xrightarrow[\pi]{\Phi} H^n(P^\cdot) \xrightarrow{H^n(f)} H^n(Q^\cdot) \xhookleftarrow{\text{via } j} {}'Z^n(Q^\cdot)$$

is the same as the composite

$$Z^n(P^\cdot) \xrightarrow{j} P^n \xrightarrow{f^n} Q^n \xrightarrow{\pi} {}'Z^n(Q^\cdot).$$

This proves our Proposition. // QED

The short exact sequences of complexes (*), i.e.,

$$0 \to Q^\cdot \xrightarrow{i} C_f^\cdot \to P^\cdot[1] \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^{n-1}(C^\cdot) \to \underset{\substack{\| \\ H^n(P^\cdot)}}{H^{n-1}(P^\cdot[1])} \xrightarrow{c_n} H^n(Q^\cdot) \to H^n(C^\cdot) \to \cdots$$

<u>Proposition</u>: For each $n \in \mathbb{Z}$, the connecting map

$$c_n : H^n(P^\cdot) \to H^n(Q^\cdot)$$

above is the same as $H^n(f)$, i.e.,

$$c_n = H^n(f).$$

<u>Proof</u>:

We have a commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \to & Q^{n-1} & \xrightarrow{i^{n-1}} & C^{n-1} & \xrightarrow{p^{n-1}} & P^n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & {}'Z^{n-1}(Q^\cdot) & \to & {}'Z^{n-1}(C^\cdot) & \to & {}'Z^n(P^\cdot) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z^n(Q^\cdot) & \to & Z^n(C^\cdot) & \to & Z^{n+1}(P^\cdot) & & \\
 & & j \downarrow & & j \downarrow & & \downarrow j & & \\
0 & \to & Q^n & \xrightarrow[i^n]{} & C^n & \xrightarrow[p^n]{} & P^{n+1} & \to & 0
\end{array}$$

Let $\quad \bar{K} = \ker\left({}^{`}Z^{n-1}(C^\bullet) \longrightarrow Z^{n+1}(P^\bullet) \right)$

and

$\qquad \bar{K}' = \operatorname{coker}\left({}^{`}Z^{n-1}(Q^\bullet) \longrightarrow Z^{n}(C^\bullet) \right).$

We have natural maps (with k, k' as in previous proof)

$$k \twoheadrightarrow \bar{K} \qquad \text{(epi)}$$
$$\bar{K}' \rightarrowtail k' \qquad \text{(mono)}.$$

The data fits into a commutative diagram

$$\begin{tikzcd}
\bar{K} \arrow[rrr] \arrow[ddd] & & & \bar{K}' \arrow[dl] \\
& k \arrow[ul] \arrow[r] \arrow[d] & k' & \\
& Z^n(P^\bullet) \arrow[r, "\varphi"] \arrow[dl, "\pi"'] & {}'Z^n(Q^\bullet) \arrow[u] & \\
H^n(P^\bullet) \arrow[rrr, "c_n"'] & & & H^n(Q^\bullet) \arrow[uuu] \arrow[ul, "\text{via } j"']
\end{tikzcd}$$

Here $\varphi : Z^n(P^\bullet) \longrightarrow {}'Z^n(Q^\bullet)$ is the connecting map of the previous Propⁿ. According to that Propⁿ φ factors as

$$\varphi = Z^n(P^\bullet) \overset{\pi}{\twoheadrightarrow} H^n(P^\bullet) \xrightarrow{H^n(f)} H^n(Q^\bullet) \xhookrightarrow{\text{via } j} {}'Z^n(Q^\bullet).$$

Since π is an epi and $H^n(Q^\bullet) \hookrightarrow {}'Z^n(Q^\bullet)$ is mono it follows that

$$c_n = H^n(f)$$

as required.

q.e.d.

~~Theorem~~ Definition: A map of complexes $f: C^\bullet \to D^\bullet$ is called a quasi-isomorphism if $H^n(f): H^n(C^\cdot) \to H^n(D^\cdot)$ is an isomorphism for every $n \in \mathbb{Z}$.

Theorem: Let $f: P^\cdot \to Q^\cdot$ be a map of complexes. Then f is an isomorphism if and only if the mapping cone C_f° of f is exact.

Proof:

We have an exact sequence (from previous Prop^n)

$$\cdots \to H^{n-2}(C^\cdot) \to H^{n-1}(P^\cdot) \xrightarrow{H^{n-1}(f)} H^{n-1}(Q^\cdot) \to H^{n-1}(C^\cdot) \to H^n(P^\cdot) \xrightarrow{H^n(f)} H^n(Q^\cdot) \to H^n(C^\cdot)$$
$$\to H^{n+1}(P^\cdot) \to \cdots$$

The Theorem follows.

q.e.d.