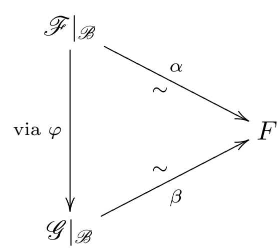
NOTES ON EXTENDING $\mathscr{B}$-SHEAVES

Suppose X is a topological space, $\mathscr{B}$ a basis for the topology on X and F a $\mathscr{B}$-sheaf. Suppose $\mathscr{F}$ is a sheaf on X such that $\mathscr{F}|_{\mathscr{B}}$ is isomorphic to F. How unique is $\mathscr{F}$? It is easy to see (see argument below) that $\mathscr{F}$ is unique up to isomorphism. However, "up to isomorphism" is a crude measure of uniqueness. There is a higher form of uniqueness that such extensions from $\mathscr{B}$ to the topology on X enjoy. In greater detail, a pair $(\mathscr{F}, \alpha)$ is called an extension of F to X if $\mathscr{F}$ is a sheaf on X and $\alpha\colon \mathscr{F}|_{\mathscr{B}} \xrightarrow{\sim} F$. A map of extensions (of F)

$$(\mathscr{F}, \alpha) \xrightarrow{\varphi} (\mathscr{G}, \beta)$$

is a map $\varphi\colon \mathscr{F} \to \mathscr{G}$ of the underlying sheaves such that the diagram below commutes.



Now that we have defined maps of extensions (of F), it makes sense to talk about isomorphisms of extensions. Here is where we see a higher form of uniqueness. Extensions of F are *unique up to unique isomorphism.* This imposes a greater rigidity on the structure than "unique up to isomorphism" does. Among other things this means that the identity map is the only automorphism of an extension.

The proof is straightforward. Indeed if U is open in X, we cover U by open subsets from $\mathscr{B}$, say by (U_i). Using $\beta^{-1}(U_i) \circ \alpha(U_i)$ we get isomorphisms

$$\mathscr{F}(U_i) \xrightarrow{\sim} \mathscr{G}(U_i).$$

If $s \in \mathscr{F}(U)$, and if for each i, t_i is the image of $s|_{U_i}$ under the above map, we see that the t_i glue to give a section $t \in \mathscr{G}(U)$. Set $\varphi(U)(s) = t$. This is clearly independent of the open cover (U_i) chosen, and one checks it gives a map of sheaves. Conversely, if there is an isomorphism $\varphi\colon \mathscr{F} \xrightarrow{\sim} \mathscr{G}$ such that the above diagram commutes, then necessarily, $\varphi(U)(s) = t$ (obvious). This nails φ as the only map such that the above diagram commutes.

In fact, a little thought shows that maps of extensions are necessarily isomorphisms of extensions, therefore even the endomorphism ring of an extension is a singleton, namely the identity map.

We saw in HW-II that if $\mathscr{B}$ has the property

$$(B_1, B_2 \in \mathscr{B}) \Rightarrow (B_1 \cap B_2 \in \mathscr{B})$$

then extensions exist. In fact extensions always exist, as you can verify yourself. You can either modify the strategy given in problems 2—6 of HW-II or you could use the strategy given in problem 7, or you could go back to basics and create an étale space $\mathscr{E}(F) \xrightarrow{\pi} X$ from the stalks of the $\mathscr{B}$-sheaves.