Hint for HW5, Problem 5

Let X be a topological space and $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ an open cover of X with I a well-ordered set. Let $\mathcal{F}$ be a sheaf on X.

Let $\partial^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ be the standard coboundary on $C^\bullet(\mathcal{U}, \mathcal{F})$, $p \geq 0$. For $p \geq 0$, let $\Sigma_I^p \subset \underbrace{I \times \cdots \times I}_{p+1}$ be the set of strictly increasing sequences. In other words $\underline{\alpha} = (\alpha_0, \ldots, \alpha_p) \in \Sigma_I^p$ iff $\alpha_0 < \alpha_1 < \cdots < \alpha_p$.

If $\underline{\alpha} \in \Sigma_I^p$ and $\underline{\beta} \in \Sigma_I^{p+1}$ we say $\underline{\beta}$ is an <u>immediate refinement</u> of $\underline{\alpha}$ if $\{\alpha_0, \ldots, \alpha_p\} \subset \{\beta_0, \ldots, \beta_{p+1}\}$. In this case there exists a unique integer $j(\underline{\alpha}, \underline{\beta})$, s.t. $0 \leq j(\underline{\alpha}, \underline{\beta}) \leq p+1$ and

$$\{\beta_0, \ldots, \beta_{p+1}\} = \{\alpha_0, \ldots, \alpha_p\} \cup \{\beta_{j(\underline{\alpha}, \underline{\beta})}\}.$$

Now suppose $\underline{\alpha} \in \Sigma_I^p$. We have a map (inclusion) $i^p_{\underline{\alpha}} : \mathcal{F}(U_{\underline{\alpha}}) \hookrightarrow C^p(\mathcal{U}, \mathcal{F})$ and a map (projection) $\pi^p_{\underline{\alpha}} : C^p(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U_{\underline{\alpha}})$.
For $\underline{\alpha} \in \Sigma_I^p$ and $\underline{\beta} \in \Sigma_I^{p+1}$ define

$$\partial^p_{\underline{\alpha}, \underline{\beta}} : \mathcal{F}(U_{\underline{\alpha}}) \longrightarrow \mathcal{F}(U_{\underline{\beta}})$$

by the formula

$$\partial^p_{\underline{\alpha}, \underline{\beta}} = \pi^{p+1}_{\underline{\beta}} \circ \partial^p \circ i^p_{\underline{\alpha}}$$

$$\left(\begin{tikzcd} C^p(\mathcal{U}, \mathcal{F}) \arrow[r, "\partial^p"] & C^{p+1}(\mathcal{U}, \mathcal{F}) \arrow[d, "\pi^{p+1}_{\underline{\beta}}"] \\ \mathcal{F}(U_{\underline{\alpha}}) \arrow[u, "i^p_{\underline{\alpha}}"] \arrow[r, "\partial^p_{\underline{\alpha}, \underline{\beta}}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{F}(U_{\underline{\beta}}) \end{tikzcd}\right)$$

First show that for $s \in \cancel{C^p(\mathcal{U}, \mathcal{F})}$ $\mathcal{F}(U_{\underline{\alpha}})$

$$\partial^p_{\underline{\alpha}, \underline{\beta}}(s) = \begin{cases} 0 & \text{if } \underline{\beta} \text{ is NOT an immediate refinement of } \underline{\alpha} \\ (-1)^{j(\underline{\alpha}, \underline{\beta})} (s|U_{\underline{\beta}}) & \text{if } \underline{\beta} \text{ is an immediate refinement of } \underline{\alpha}. \end{cases}$$

Next suppose I has a largest index, say ∞. Let $J = I \setminus \{\infty\}$, and set $\mathcal{U}_J = \{U_\alpha\}_{\alpha \in J}$, $U_J = \bigcup_{\alpha \in J} U_\alpha$, $U^* = U_J \cap U_\infty$.
If $V_\alpha = U_\infty \cap U_\alpha$, $\alpha \in J$, then $\mathcal{V} = (V_\alpha)_{\alpha \in J}$ is a cover of U^*.

Should be "script U upper star" and not "script V" here.

It is easy to see that we have a map of complexes

$$\sigma = (\sigma^p) : \mathcal{C}^\bullet(\mathcal{U}_J, \mathcal{F}|_{\mathcal{U}_J}) \longrightarrow \mathcal{C}^\bullet(\mathcal{U}^*, \mathcal{F}|_{\mathcal{U}^*})$$

given by "restriction".

Now consider the anti-commuting double complex

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U_\infty) \arrow[r] & \mathcal{C}^0(\mathcal{U}^*, \mathcal{F}|_{\mathcal{U}^*}) \arrow[r] & \mathcal{C}^1(\mathcal{U}^*, \mathcal{F}|_{\mathcal{U}^*}) \arrow[r] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U}^*, \mathcal{F}|_{\mathcal{U}^*}) \arrow[r] & \cdots \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathcal{C}^0(\mathcal{U}_J, \mathcal{F}|_{U_J}) \arrow[r] \arrow[u, "-\sigma^0"'] & \mathcal{C}^1(\mathcal{U}_J, \mathcal{F}|_{U_J}) \arrow[r] \arrow[u, "\sigma^1"'] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U}_J, \mathcal{F}|_{U_J}) \arrow[r] \arrow[u, "(-1)^{p+1}\sigma^p"'] & \cdots
\end{tikzcd}$$

where $\mathcal{F}(U_\infty)$ is in the ~~(-1,-1)th~~ $(-1,1)^{th}$ spot, and $\mathcal{C}^0(\mathcal{U}_J, \mathcal{F}|_{U_J})$ is in the $(0,0)^{th}$ spot and $\mathcal{F}(U_\infty) \longrightarrow \mathcal{C}^0(\mathcal{U}^*, \mathcal{F}|_{\mathcal{U}^*})$ is given by restriction. Check that the total complex of this anti-commuting complex is $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ — you will find it helpful to use the formula for $\partial^p_{\underline{\alpha}, \underline{\beta}}$ given in the previous page.

Apply this to the problem on hand by proving

$$K^\bullet_\infty(\underline{t}, A) = K^\bullet_\infty(t_1, A) \otimes K^\bullet_\infty(t_2, A) \otimes \cdots \otimes K^\bullet_\infty(t_n, A)$$
$$= K^\bullet_\infty((t_1, \ldots, t_{n-1}), A) \otimes K^\bullet_\infty(t_n, A)$$

and by checking that $K^\bullet(t_i, A)$ is $0 \to A \to A_{t_i} \to 0$ where A is placed in degree 0. Now use induction, i.e. assume that the assertion of Prop 5 is true for $K^\bullet_\infty((t_1, \ldots, t_{n-1}), A)$. You will of course have to check that the assertion is true at some initial stage ($n=1$ or $n=2$).

<u>Corrections & Comments</u>: 1. In problem 7, $W = \operatorname{Spec} S$ and not $\operatorname{Spec} R$.

2. In problem 8, it should be $H^{n-1}(\mathcal{U}, \mathcal{O}_U)$ and NOT $H^{n-1}(U, \mathcal{O}_U)$ (i.e. replace U in $H^{n-1}(U, -)$ by $\mathcal{U}$).