HW7

Throughout this HW, X is a *locally compact topological space*. Recall that by a *sheaf of commutative rings*[1] $\mathscr{A}$ on X, it is understood that the sections over any open set is a commutative ring with 1 and that the restriction maps respect the ring structure and map 1 to 1. A sheaf of $\mathscr{A}$-modules on X is a sheaf $\mathscr{E}$ such that $\Gamma(U, \mathscr{E})$ carries the structure of a unital $\Gamma(U, \mathscr{A})$-module for each open set U, and the restriction maps r_{UV} satisfy (for V an open subset of U):

$$r_{UV}(ae) = r_{UV}(a)r_{UV}(e) \qquad (a \in \Gamma(U, \mathscr{A}),\, e \in \Gamma(U, \mathscr{E})).$$

(1) Here is a Urysohn type result for soft sheaves. Let $\mathscr{A}$ be a soft sheaf of rings on X, K a compact subset of X and L a compact neighbourhood of K (this means that $K \subset L^\circ$). Show that there exists $\sigma \in \Gamma(X, \mathscr{A})$ such that $\sigma = 1$ in an open neighbourhood of K and $\sigma = 0$ on the complement of L°.

(2) Let $\mathscr{A}$ be a soft sheaf of rings on X and $\mathscr{E}$ a sheaf of $\mathscr{A}$-modules. Show that $\mathscr{A}$ is soft.

(3) (Local Operators) Let $\mathscr{S}$ and $\mathscr{T}$ be sheaves on the locally compact space X and $D\colon \Gamma_c(X\, \mathscr{S}) \to \Gamma_c(X, \mathscr{T})$ a linear map which satisfies the following condition

$$\operatorname{Supp} D(f) \subset \operatorname{Supp}(f) \qquad \text{for all } f \in \Gamma_c(X, \mathscr{S}).$$

If $\mathscr{S}$ is soft then show that there exists a unique morphism of sheaves $d\colon \mathscr{S} \to \mathscr{T}$ which induces D on the level of global sections.

(4) Let $i\colon Z \hookrightarrow X$ be the inclusion of a closed subspace. Define $i^!\colon \mathrm{Sh}_X \to \mathrm{Sh}_Z$ by

$$i^!\mathscr{F} = i^{-1}\underline{\Gamma_Z}(\mathscr{F}) \qquad (\mathscr{F} \in \mathrm{Sh}_X).$$

Show that $i^!$ is right adjoint to i_*.

(5) For $h\colon W \to X$ the inclusion of a *locally closed* subset W of X into X, and a sheaf $\mathscr{G}$ on W, define $h_!(\mathscr{G}) \in \mathrm{Sh}_X$ by

$$\Gamma(U, h_!\mathscr{G}) := \{s \in \Gamma(W \cap U) \,|\, \operatorname{Supp}(s) \text{ is closed relative to } U\}.$$

The restriction maps are the ones induced from $h_*\mathscr{E}$. Note that $h_!\mathscr{E}$ is a subsheaf of $h_*\mathscr{E}$ and let

$$h_!\mathscr{E} \to h_*\mathscr{E}$$

be the canonical map

(a) Show that $h_!$ is exact.

(b) Show that if $Z \xrightarrow{j} W$ is a locally closed inclusion then $(hj)_! = h_! j_!$.

[1] Often shortened to *sheaf of rings* in this course.

(c) Show that $h_!$ has a right adjoint $h^!$. [Hint: Note that h can be written as the composite of a closed inclusion and an open inclusion. What is $h_!$ when h is closed?]